This work has arisen from lecture courses given by the authors on important topics within functional analysis. The authors, who are all leading researchers, give introductions to their subjects at a level ideal for beginning graduate students, as well as others interested in the subject. The collection has been carefully edited to form a coherent and accessible introduction to current research topics.

The first part of the book, by Professor Dales, introduces the general theory of Banach algebras, which serves as a background to the remaining material. Dr Willis then studies a centrally important Banach algebra, the group algebra of a locally compact group. The remaining chapters are devoted to Banach algebras of operators on Banach spaces: Professor Eschmeier gives all the background for the exciting topic of invariant subspaces of operators, and discusses some key open problems; Dr Laursen and Professor Aiena discuss local spectral theory for operators, leading into Fredholm theory.

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INTRODUCTION TO BANACH ALGEBRAS, OPERATORS, AND HARMONIC ANALYSIS

H. GARTH DALES
University of Leeds, UK

PIETRO AIENA
Università degli Studi, Palermo, Italy

JÖRG ESCHMEIER
Universität des Saarlandes, Saarbrücken

KJELD LAURSEN
University of Copenhagen, Denmark

GEORGE A. WILLIS
University of Newcastle, New South Wales, Australia

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Contents

Preface

This volume is based on a collection of lectures intended for graduate students and others with a basic knowledge of functional analysis. It surveys several areas of current research interest, and is designed to be suitable preparatory reading for those embarking on graduate work. The volume consists of five parts, which are based on separate sets of lectures, each by different authors. Each part provides an overview of the subject that will also be useful to mathematicians working in related areas. The chapters were originally presented as lectures at instructional conferences for graduate students, and we have maintained the styles of these lectures.

The sets of lectures are an introduction to their subjects, intended to convey the flavour of certain topics, and to give some basic definitions and motivating examples: they are certainly not comprehensive accounts. References are given to sources in the literature where more details can be found.

The chapters in Part I are by H. G. Dales. These are an introduction to the general theory of Banach algebras, and a description of the most important examples: $\mathcal{B}(E)$, the algebra of all bounded linear operators on a Banach space E; $L^1(G)$, the group algebra of a locally compact group G, taken with the convolution product; commutative Banach algebras, including Banach algebras of functions on compact sets in $\mathbb{C}$ and radical Banach algebras. Chapters 3–6 cover Gelfand theory for commutative Banach algebras, the analytic functional calculus, and, in a chapter on 'automatic continuity', the lovely results that show the intimate connection between the algebraic and topological structures of a Banach algebra. Chapters 6 and 7 are an introduction to the cohomology theory of Banach algebras, at present a very active area of research; we concentrate on the basic structure, that of derivations into modules.

The chapters in Part II, by G. A. Willis, develop the theory of one of the examples discussed by Dales: these are the group algebras $L^1(G)$. Chapters 8 and 9 give a description of locally compact groups G and their structure theory, and then describe the algebras $L^1(G)$, and the related measure algebra $M(G)$, as a Banach algebra. The Gelfand theory for general commutative Banach algebras, as described by Dales, becomes Fourier transform theory in the special case of the algebras $L^1(G)$. In Chapter 10, Willis discusses compact groups, abelian groups, and free groups, and then, in Chapter 11, moves to a very important class, that of amenable groups: many characterizations of amenability arise in diverse areas of mathematics. Willis then expands a notion from Part I by discussing the automatic continuity of linear maps from group algebras.

Parts III–V of this book develop the theory of another example mentioned by Dales: this is the algebra $\mathcal{B}(E)$ for a Banach space E. However, they also concentrate on the properties of single operators of various types within $\mathcal{B}(E)$.

A seminal question in functional analysis is the 'invariant subspace problem'. Let E be a Banach space, and let $T \in \mathcal{B}(E)$. A closed subspace F of E is invariant for T if $Ty \in F \ (y \in F)$; F is trivial if $F = \{0\}$ or $F = E$. Does such an operator T always have a non-trivial invariant subspace? A positive answer to this question in the case where E is finite-dimensional (of dimension at least 2) is the first step in the structure theory of matrices. The question for Banach spaces has been the spur for a huge amount of research in operator theory since the question was first raised in the 1930s. The question is still open in the case where E is a Hilbert space – this is one of the great problems of our subject – but counter-examples are known when E is an arbitrary Banach space. Nevertheless, there are many positive results for operators $T \in \mathcal{B}(E)$ which belong to a special class.

The chapters in Part III, by J. Eschmeier, discuss in particular one very important technique for establishing positive results: it descends from original work of Scott Brown in 1978. One class of operators considered is that of subdecomposable operators. Part III concludes with remarks about the extensions, mainly due to the author, of the positive results to n-tuples of commuting operators.

As explained by Dales, every element a of a Banach algebra has a spectrum, called $\sigma(a)$; this is a non-empty, compact subset of the complex plane $\mathbb{C}$. In particular, each operator $T \in \mathcal{B}(E)$ has such a spectrum, $\sigma(T)$. In the case where E is finite-dimensional, $\sigma(T)$ is just the set of eigenvalues of T. The notion of the spectrum for a general operator T is at the heart of the remaining chapters, by K. B. Laursen and P. Aiena.

Laursen discusses the spectral theory of operators in several different classes; these include in particular the decomposable operators, which were also introduced by Eschmeier. We understand the nature of an operator T by looking at the decomposition of $\sigma(T)$ into subsets with special properties and also at special closed subspaces of E on which T acts 'in a nice way'. In particular, Laursen discusses super-decomposable and generalized scalar operators. In Chapter 25, Laursen relates his description to notions introduced by Dales by discussing when multiplication operators on commutative Banach algebras have the various properties that he has introduced. A valuable appendix to Part IV sketches the background theory, involving distributions, to the 'functional model' of Albrecht and Eschmeier that is the natural setting for many of the duality results that have been obtained.

The final chapters, those of P. Aiena, are closely related to those of Laursen. The basic examples of the decomposable operators of Eschmeier's and Laursen's chapters are compact operators on a Banach space and normal operators on a Hilbert space. It is natural to study the decomposable operators which have similar properties to those of these important specific examples: we are led to the class of 'Fredholm operators' and related classes, a main topic of Aiena's lectures.

In these chapters, we see again, from a different perspective, some of the key ideas – decomposition of the spectrum, invariant subspace, single-valued extension property, actions of analytic functions, divisible subspaces – that have featured in earlier chapters. The final chapter by Aiena summarizes recent work of the author and others.

The lectures on which this book is based were given at two conferences. The first was held in Mussomeli, Sicily, from 22 to 29 September 1999. We are very grateful to Dr Gianluigi Oliveri, who organized this conference, and to the Associazone Culturale Archimede of Sicily who sponsored it. The lectures given at this conference were those of Dales, Eschmeier, Laursen, and Aiena. The second conference was held at the Sadar Patel University, Vallabh Vidyanagar, Gujarat, India, from 8 to 15 January 2002. We are very grateful to Professor Subhashbhai Bhatt and Dr Haresh Dedania for organizing this conference, and to the Indian Board for Higher Mathematics and to the London Mathematical Society, who supported the conference financially. The lectures given in Gujarat were those of Dales, Willis, and Laursen.

As we said, the original lectures were intended for graduate students and others with a basic knowledge of functional analysis and with a background in complex analysis and algebra typical of a first degree in mathematics. In both cases the students were enthusiastic and helpful; their suggestions led to many improvements in the exposition, and we are grateful to them for this.

In fact, the actual lectures as given did not include all that is written down here: modest additions have been made subsequently. There is more in a 'lecture' than can easily be absorbed in one hour. However, we have maintained the fairly informal style of the lecture theatre. At various points, the reader is invited to check statements that are made: these are all routine, and follow in a few lines from facts given in the lectures. There are also exercises at the end of each lecture; the answers to all the exercises are contained in the references that are specified. We hope that readers will work through the exercises as a step towards the gaining of familiarity with the subject.

There are various cross-references between the sets of lectures; indeed topics of one set of lectures often reappear, perhaps in a different guise, in other lectures. All the book depends somewhat on the first six chapters, and Part V follows from Part IV. However, otherwise the various parts of the book can be read independently. The references to each set of lectures are contained at the end of the relevant part of this book, and not at the end of the whole book. However, there are two indices for the whole book at the end (pp. 319–326): these are the symbol index and the index of terms.

Some attempt has been made to make the notation consistent between the various sets of lectures, but we have not always achieved this; we give a resumé of some standard notations at the end of this preface.

We very much enjoyed giving the original lectures and discussing the theory and associated examples in the classes that were given in the same week as the lectures. We hope that you enjoy reading them and, especially, working through the examples.

In rather more detail, we expect the reader to be familiar with the following topics:

- the definition of a Banach space and a locally convex space, weak topologies on dual spaces;
- standard theorems of functional analysis such as the Hahn–Banach theorem, closed graph theorem, open mapping theorem, and uniform boundedness theorem;
- the theory of bounded linear operators on a Banach space, duals of such operators, compact operators;
- the elementary theory of Hilbert spaces;
- undergraduate complex analysis, including Liouville's theorem;
- undergraduate algebra, including the theory of ideals, modules, and homomorphisms.

Throughout we adopt the following notation:
$\mathbb{N} = \{1, 2, 3, \ldots\}$;
$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$;
$\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$;
$\mathbb{Q}$ is the field of rational numbers;

$\mathbb{R}$ is the field of real numbers;
$\mathbb{C}$ is the field of complex numbers;
$\mathbb{I} = [0, 1]$;
$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$;
$\mathbb{D}(z; r) = \{w \in \mathbb{C} : |w - z| < r\}$;
$\mathbb{D} = \mathbb{D}(0; 1)$;
Z is the coordinate functional on $\mathbb{C}$, or on a subset of $\mathbb{C}$;
E' is the dual space of a topological linear space E;
$E_{[1]}$ is the closed unit ball of a Banach space E;
$[x, y]$ is the inner product of $x, y \in H$, where H is a Hilbert space;
$\langle x, \lambda \rangle$ is the action of $\lambda \in E'$ on $x \in E$, where E is a Banach space.

H. G. D., Leeds