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Introduction to Möbius Differential Geometry

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Introduction

Over the past two decades, the geometry of surfaces and, more generally, submanifolds in Möbius geometry has (re)gained popularity. It was probably T. Willmore's 1965 conjecture [306] that stimulated this increased interest: Many geometers have worked on this conjecture, and in the course of this work it turned out (see for example [67] and [40]) that the Willmore conjecture is in fact a problem for surfaces in Möbius geometry and that the corresponding local theory was already developed by the classical geometers (cf., [218]). A crucial classical reference was [29]; however, it may not be very easy to obtain and, once found, may not be very easy to read, especially for non-German—speaking colleagues.

A similar story could be told about the recent developments on isothermic surfaces — here, it was the relation with the theory of integrable systems, first pointed out in [71], that made the topic popular again; also in this case [29] turned out to be a treasure trove, but many more results are scattered in the classical literature.

The present book has a twofold purpose:

- It aims to provide the reader with a solid background in the Möbius geometry of surfaces and, more generally, submanifolds.
- It tries to introduce the reader to the fantastically rich world of classical (Möbius) differential geometry.

The author also hopes that the book can lead a graduate student, or any newcomer to the field, to recent research results.

Before going into details, the reader's attention shall be pointed to three[1)] other textbooks in the field. To the author's knowledge these are the only books that are substantially concerned with the Möbius geometry of surfaces or submanifolds:

1. W. Blaschke [29]: Currently, this book is a standard reference for the geometry of surfaces in 3-dimensional Möbius geometry where many fundamental facts (including those concerning surface classes of current interest) can be found. Möbius geometry is treated as a subgeometry of Lie sphere geometry.
2. T. Takasu [275]: Nobody seems to know this book; like [29], it is in German, and the presented results are similar to those in Blaschke's book — however, Möbius geometry is treated independently.

[1)] Apparently there is another classical book [95] by P. C. Delens on the subject that the present author was not able to obtain so far.

3. M. Akivis and V. Goldberg [4]: This is a modern account of the theory, generalizing many results to higher dimension and/or codimension. Also, different signatures are considered, which is relevant for applications in physics. In this book the authors also discuss nonflat conformal structures; almost Grassmann structures, a certain type of Cartan geometry, are considered as a closely related topic (compare with the survey [7]).

Besides these three textbooks, there is a book with a collection of (partially introductory) articles on Möbius or conformal differential geometry [170]. Here, Möbius geometry is mainly approached from a Riemannian viewpoint (see, for example, J. Lafontaine's article [173]), which is similar to the way it is touched upon in many textbooks on differential geometry but in much greater detail and including a description of the projective model.

A more general approach to Möbius geometry than the one presented in this book may be found in [254], see also the recent paper [50]: There, Möbius geometry is treated as an example of a Cartan geometry.

I.1 Möbius geometry: models and applications

In Möbius geometry there is an angle measurement but, in contrast to Euclidean geometry, no measurement of distances. Thus Möbius geometry of surfaces can be considered as the geometry of surfaces in an ambient space that is equipped with a conformal class of metrics but does not carry a distinguished metric. Or, taking the point of view of F. Klein [160], one can describe the Möbius geometry of surfaces as the study of those properties of surfaces in the (conformal) n-sphere S^n that are invariant under Möbius (conformal) transformations of S^n. Here, "Möbius transformation" means a transformation that preserves (hyper)spheres in S^n, where a hypersphere can be understood as the (transversal) intersection of an affine hyperplane in Euclidean space $\mathbb{R}^{n+1}$ with $S^n \subset \mathbb{R}^{n+1}$.

Note that the group of Möbius transformations of $S^n \cong \mathbb{R}^n \cup \{\infty\}$ is generated by inversions, that is, by reflections in hyperspheres in S^n.

The lack of length measurement in Möbius differential geometry has interesting consequences; for example, from the point of view of Möbius geometry, the planes (as spheres that contain ∞) and (round) spheres of Euclidean 3-space are not distinguished any more.

At this point we can already see how Euclidean geometry is obtained as a subgeometry of Möbius geometry when taking the Klein point of view: The group of Möbius transformations is generated by inversions; by restricting to reflections in planes (as "special" spheres), one obtains the group of isometries of Euclidean space.

Also, the usual differential geometric invariants of surfaces in space lose their meaning; for example, the notion of an induced metric on a surface as well as the notion of curvature lose their meanings as the above example of spheres and planes in $\mathbb{R}^3$ illustrates — the reason is, of course, that these notions are defined using the metric of the ambient space. But the situation is less hopeless than one might expect at first: Besides the angle measurement (conformal structure) that a surface or submanifold inherits from the ambient space, there are conformal invariants that encode some of the curvature behavior of a surface. However, it is rather complicated and unnatural to consider submanifolds in Möbius geometry as submanifolds of a Riemannian (or Euclidean) space and then to extract those properties that remain invariant under the larger symmetry group of conformal transformations.[2)]

I.1.1 Models of Möbius geometry. Instead, one can describe Möbius geometry in terms of certain "models" where hyperspheres (as a second type of "elements" in Möbius geometry, besides points) and the action of the Möbius group are described with more ease. For example, we can describe hyperspheres in $S^n \subset \mathbb{R}^{n+1}$ by linear equations instead of quadratic equations as in $S^n \cong \mathbb{R}^n \cup \{\infty\}$ — however, it is still unpleasant to describe the Möbius group acting on $S^n \subset \mathbb{R}^{n+1}$.

Let us consider another example to clarify this idea: In order to do hyperbolic geometry, it is of great help to consider a suitable model of the hyperbolic ambient space, that is, a model that is in some way adapted to the type of problems that one deals with. One possibility would be to choose the Klein model of hyperbolic space where H^n is implanted into projective n-space $\mathbb{R}P^n$; then the hyperbolic motions become projective transformations that map H^n to itself, that is, projective transformations that preserve the infinity boundary $\partial_\infty H^n$ of the hyperbolic space, and hyperplanes become the intersection of projective hyperplanes in $\mathbb{R}P^n$ with H^n. In this model, for example, it is obvious that the Euclidean Parallel Postulate does not hold in hyperbolic geometry. Another possibility would be to consider H^n as one of the connected components of the 2-sheeted hyperboloid

$$\{y \in \mathbb{R}^{n+1}_1 \mid \langle y, y \rangle = -1\}$$

in Minkowski $(n+1)$-space $\mathbb{R}^{n+1}_1 = (\mathbb{R}^{n+1}, \langle .,. \rangle)$. In this model[3)] it is rather simple to do differential geometry. A third possibility, which we will come

[2)] This will become obvious when we take this viewpoint in the "Preliminaries" chapter that is meant as an introduction for those readers who have a background in Riemannian geometry but are new to Möbius geometry.

[3)] It is a matter of taste whether one wants to consider this as a different *model* for hyperbolic geometry: Of course there is a simple way to identify this "model" with the Klein model; it is just a convenient choice of homogeneous coordinates for the Klein model after equipping the coordinate $\mathbb{R}^{n+1}$ with a scalar product.

across in the present text more often, will be to use the Poincaré (half-space or ball) model of hyperbolic geometry, where H^n is considered as a subspace of the conformal n-sphere. In this model, the hyperbolic hyperplanes become (the intersection of $H^n \subset S^n$ with) hyperspheres that intersect the infinity boundary $\partial_\infty H^n \subset S^n$ of the hyperbolic space orthogonally and reflections in these hyperplanes are inversions.

In this book we will elaborate three and a half models[4)] for Möbius geometry.

I.1.2 The projective model. This is the approach that the classical differential geometers used, and it is still the model that is used in many modern publications in the field. In fact, it can be considered as *the* model, because all other "models" can be derived from it.[5)] A comprehensive treatment of this model can be found in the book by M. Akivis and V. Goldberg [4], which the reader is also encouraged to consult; our discussions in Chapter 1 will follow instead the lines of W. Blaschke's aforementioned book [29].

To obtain this projective model for Möbius geometry the ambient (conformal) n-sphere S^n is implanted into the projective $(n+1)$-space $\mathbb{R}P^{n+1}$. In this way hyperspheres will be described as the intersection of projective hyperplanes with $S^n \subset \mathbb{R}P^{n+1}$, in a similar way as discussed above, and Möbius transformations will be projective transformations that preserve S^n as an "absolute quadric" so that, in homogeneous coordinates, the action of the Möbius group is linear. This is probably the most important reason for describing Möbius geometry as a subgeometry of projective geometry.[6)]

I.1.3 The quaternionic approach. Here the idea is to generalize the description of Möbius transformations of $S^2 \cong \mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$ as fractional linear transformations from complex analysis to higher dimensions. For dimensions 3 and 4, this can be done by using quaternions — this approach can be traced back to (at least) E. Study's work [262]. Recently, the use of the quaternionic model for Möbius geometry has provided remarkable progress in the global Möbius geometry of surfaces [113].

One of the main difficulties in establishing this quaternionic approach to Möbius geometry is the noncommutativity of the field of quaternions. Besides that, it is rather seamless to carry over much of the complex theory.

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4) As mentioned above, it is, to a certain extent, a matter of taste of what one considers as different "models" and what one considers to be just different incarnations of one model.

5) Therefore, one could consider this model to be the only model and all other "models" to be different representations or refinements of it; however, this is not the author's viewpoint.

6) In [160], F. Klein undertakes it to describe many geometries as subgeometries of projective geometry and, in this way, to bring order to the variety of geometries (compare with [254]).

An important issue will be the description of (hyper)spheres in this model. For this it is convenient to link the quaternionic model to the projective model by using quaternionic Hermitian forms; for the important case of 2-spheres, there will be a second description based on Möbius involutions.

I.1.4 The Clifford algebra approach. After describing Möbius geometry as a subgeometry of projective geometry it is rather natural to use the Clifford algebra of the $(n+2)$-dimensional space of homogeneous coordinates of the host projective space, equipped with a Minkowski scalar product, in order to describe geometric objects (cf., [155] and [154]) — just notice that the development of this algebra, initiated by H. Grassmann in [130] and [133] and by W. K. Clifford in [77], was originally motivated by geometry, as its original name, "geometric algebra," suggests. For example, the description of spheres of any codimension becomes extremely simple using this approach. Our discussions in Chapter 6 on this topic were motivated by an incomplete 1979 manuscript by W. Fiechte [117].

More common is an enhancement[7)] of this Clifford algebra model by writing the elements of the Clifford algebra of the coordinate Minkowski space $\mathbb{R}^{n+2}_1$ of the classical model as 2×2 matrices with entries from the Clifford algebra of the Euclidean n-space $\mathbb{R}^n$. This approach can be traced back to a paper by K. Vahlen [288]; see also [2] and [3] by L. Ahlfors.

In some sense this model can also be understood as an enhancement of the quaternionic model (for 3-dimensional Möbius geometry): Möbius transformations, written as Clifford algebra 2×2 matrices, act on the conformal n-sphere $S^n \cong \mathbb{R}^n \cup \{\infty\}$ by fractional linear transformations.[8)] The description of Möbius transformations by 2×2 matrices makes this approach to Möbius geometry particularly well suited for the discussion of the geometry of "point pair maps" that arise, for example, in the theory of isothermic surfaces as developed in the excellent paper by F. Burstall [47].

I.1.5 Applications. The description of each model in turn is complemented by a discussion of applications to specific problems in Möbius differential geometry. The choice of these problems is certainly influenced by the author's preferences, his interest and expertise — however, the author hopes to have chosen applications that are of interest to a wider audience and that can lead the reader to current research topics.

Conformally flat hypersurfaces are discussed in Chapter 2. Here we already touch on various topics that will reappear in another context or in more generality later; in particular, we will come across curved flats, a particularly

[7)] Thus we count the two descriptions as one and a half "models."

[8)] This matrix representation of the Clifford algebra $\mathcal{A}\mathbb{R}^{n+2}_1$ corresponds to its "conformal split" (see [154]); this choice of splitting provides a notion of stereographic projection (cf., [47]).

simple type of integrable system, and Guichard nets, that is, a certain type of triply orthogonal system.

Willmore surfaces are touched on as a solution of a Möbius geometric problem that we will refer to as "Blaschke's problem." In this text we will cover only basic material on Willmore surfaces; in particular, we will not get into the large body of results concerning the Willmore conjecture.

Isothermic surfaces will appear as another solution of Blaschke's problem. A first encounter of the rich transformation theory of isothermic surfaces is also given in Chapter 3, but a comprehensive discussion is postponed until the quaternionic model is available in Chapter 5; in the last two sections of Chapter 8 we will reconsider isothermic surfaces and show how to generalize the results from Chapter 5 to arbitrary codimension. We will make contact with the integrable systems approach to isothermic surfaces, and we will discuss a notion of discrete isothermic nets in the last section of Chapter 5.

Orthogonal systems will be discussed in Chapter 8. We will generalize the notion of triply orthogonal systems to m-orthogonal systems in the conformal n-sphere and discuss their Ribaucour transformations, both smooth and discrete. The discussion of discrete orthogonal nets is somewhat more geometrical than that of discrete isothermic nets, and it demonstrates the interplay of geometry and the Clifford algebra formalism very nicely.

I.1.6 Integrable systems. A referee of the present text very rightly made the remark that "the integrable systems are not far beneath the surface in the current text." This fact shall not be concealed: As already mentioned above, conformally flat hypersurfaces as well as isothermic surfaces are related to a particularly simple type of integrable system in a symmetric space, "curved flats," that were introduced by D. Ferus and F. Pedit in [112].

However, the corresponding material is scattered in the text, and we will not follow up on any implications of the respective integrable systems descriptions but content ourselves by introducing the spectral parameter that identifies the geometry as integrable. Instead we will discuss the geometry of the spectral parameter in more detail:

Conformally flat hypersurfaces in S^4 are related to curved flats in the space of circles, and the corresponding spectral parameter can already be found in C. Guichard's work [136]. In this case, a curved flat describes a circle's worth of conformally flat hypersurfaces, and the associated family of curved flats yields a 1-parameter family of such cyclic systems with conformally flat orthogonal hypersurfaces.

Isothermic surfaces are related to curved flats in the space of point pairs in the conformal 3-sphere S^3 (or, more generally, in S^n), and the existence of the corresponding spectral parameter will turn out to be intimately related to the conformal deformability of isothermic surfaces [62] and to the Calapso transformation, see [53] and [55]. In fact, the associated family of curved flats (Darboux pairs of isothermic surfaces) yields one of Bianchi's permutability theorems [20] that intertwines the Christoffel, Darboux, and Calapso transformations of an isothermic surface.

For more information about the respective integrable systems approaches, the reader shall be referred to [142] and to F. Burstall's paper [47].

I.1.7 Discrete net theory. More recently, discrete net theory has become a field of active research. In the author's opinion, discrete net theory, as it is discussed in the text, is of interest for various reasons: An obvious reason may be the application of the theory in computer graphics and experimental mathematics; however, the author thinks that it is also very interesting for methodological reasons — the proofs of "analogous" results in corresponding smooth and discrete theories are usually rather different. While proofs in (smooth) differential geometry are often very computational in nature, the proofs of the corresponding discrete results may be done by purely (elementary) geometric arguments. In this way proofs from discrete net theory sometimes resemble the proofs of the classical geometers when they applied geometric arguments to "infinitesimal" quantities (cf., [70]).

As already mentioned above, we will discuss two discrete theories, one of which is a special case — with more structure — of the other:

Discrete isothermic nets will be discussed in the last section of Chapter 5. We will see that much of the theory of smooth isothermic surfaces can be carried over to the discrete setup; in fact, many proofs can be carried over directly when using the "correct" discrete version — discrete quantities usually carry more information than their smooth versions. In this way, the analogous smooth and discrete theories can motivate and inspire each other.[9] Note that the fact that computations can be carried over from one setup to the other so seamlessly relies on using the quaternionic model for Möbius geometry.[10]

Discrete orthogonal nets will be treated more comprehensively, in Chapter 8. This topic demonstrates nicely the interplay of analytic and geometric methods in discrete net theory, as well as the interplay of algebra and geometry in the Clifford algebra model. In this way, its presentation serves a twofold purpose. A highlight of the presentation shall be the discrete analog of Bianchi's permutability theorem for the Ribaucour transformation of

[9] Despite the order of presentation, some of the proofs on smooth isothermic surfaces in the present text were obtained from their discrete counterparts.

[10] Or, equally, the Vahlen matrix approach should work just as well.

orthogonal systems.

I.1.8 Symmetry-breaking. Finally, the author would like to draw the reader's attention to a phenomenon that he considers to be rather interesting and that appears at various places in the text. First note that the metric (hyperbolic, Euclidean, and spherical) geometries are subgeometries of Möbius geometry.[11)] Now, imposing two Möbius geometric conditions on, say, a surface may break the symmetry of the problem and yield a characterization of the corresponding surface class in terms of a (metric) subgeometry of Möbius geometry. The most prominent example for such a symmetry-breaking may be Thomsen's theorem, which we discuss in Chapter 3: A surface that is Willmore and isothermic at the same time is (Möbius equivalent to) a minimal surface in some space of constant curvature. Other examples are Guichard cyclic systems, which turn out to come from parallel Weingarten surfaces in space forms, and isothermic or Willmore channel surfaces, which are Möbius equivalent to surfaces of revolution, cylinders, or cones in Euclidean geometry.

I.2 Philosophy and style

In the author's opinion, geometry describes certain aspects of an ideal world where geometric configurations and objects "live." In order to describe that ideal world, one needs to use some language or model — that may change even though the described objects or facts remain the same. Thus a geometer may choose from a variety of possibilities when carrying out his research or presenting his results to colleagues or students. In this choice he may be led by different motivations: sense of beauty, curiosity, pragmatism, ideology, ignorance, and so on.

I.2.1 Methodology. This book shall be an advertisement for "methodological pluralism": It will provide the reader with three and a half "models" for Möbius (differential) geometry that may be used to formulate geometric facts. In order to compare the effect of choosing different models, the reader may compare the treatment of isothermic surfaces — in terms of the classical projective model in Chapter 3, in terms of the quaternionic approach in Chapter 5, and in terms of the Vahlen matrix setup in Chapter 8. The author hopes that the chance of comparing these different descriptions of the geometry of isothermic surfaces compensates the reader for the repetitions caused by the multiple treatment.

However, we will also pursue this program of "methodological pluralism" in the details. Very often we will (because of the author's pragmatism, sense

[11)] For the hyperbolic geometry, we already touched on this when discussing the Poincaré model of hyperbolic geometry above.

of beauty, or ignorance) use Cartan's method of moving frames to describe the geometry of surfaces or hypersurfaces. However, we will always try to find an "adapted frame" that fits the geometric situation well. This notion of an adapted frame, which is central to many computations in the text, will change depending on the context. The discussion of isothermic and Willmore channel surfaces in the last section of Chapter 3 may serve as a typical example: Depending on the viewpoint we wish to take, we will use frames that are, to an attuned degree, adapted to the surface or to the enveloped sphere curve, respectively.

I.2.2 Style. Note that a "model" should always be distinguished from what it describes, and that it is unlikely that a description using a model will be optimal in any sense. This is even more the case as long as an author has not stopped working and learning about his topic — here is where the ignorance issue comes in. Therefore, the present text does not claim to provide the optimal description of a subject, and the reader is warmly invited to figure out better ways to think about, say, isothermic surfaces. However, there is another issue related to the aim of this book to make the classical literature more accessible to the reader: In this text we will try to adopt certain habits of the classical authors and make a compromise between modern technology and classical phrasing; in this way a reader may be better prepared to study the classical literature — which is sometimes not very easy to access.

I.2.3 Prerequisites. There is some background material that the reader is expected to be familiar with: some basics in semi-Riemannian geometry, on Lie groups and homogeneous spaces, and some vector bundle geometry. All the needed background material can be found in the excellent textbook [209]. Also, the reader shall be referred to [189], which is a treasure trove for algebraic (and historic) background material, in particular on Clifford algebras and quaternions.

In the "Preliminaries" chapter we will summarize some material on conformal differential geometry from the Riemannian point of view, mainly following ideas from [173]. It is meant to be a preparation for those readers who have some background in Riemannian geometry but who are new to conformal differential geometry; the author took this approach when giving a course for graduate students at TU Berlin in the winter of 1999—2000. This chapter also serves to collect some formulas and notions for later reference, and a discussion of certain conformal invariants of surfaces in the conformal 3-sphere is provided.[12)]

[12)] However, it shall be pointed out that this chapter is not meant as an introduction to nonflat conformal differential geometry; for this the reader is referred to [254] and [50].

I.2.4 Further reading. At the end of the book the author gives a selection of references for further reading, with comments.

Also, the list of references contains many more references than are cited in the text: In particular, many classical references are provided to facilitate the search for additional literature — in the author's experience it can be rather difficult to locate relevant (classical) references. For the reader's convenience the coordinates of reviews are provided where the author was able to locate a review, and some cross-references to occurrences of a reference in the text are compiled into the Bibliography.[13]

I.3 Acknowledgments

Many people have contributed, in one way or another, to this book — more than I could possibly mention here. However, I would like to express my gratitude to at least some of them.

First, I would like to thank my teacher, Ulrich Pinkall, who got me interested in Möbius differential geometry in the first place. For helpful discussions or suggestions concerning the covered topics, for constructive criticism and other helpful feedback on the manuscript, or for some other kind of support I warmly thank the following friends and/or colleagues: Alexander Bobenko, Christoph Bohle, Fran Burstall, Susanne Hannappel, Gary Jensen, Catherine McCune, Patrick McDonough, Emilio Musso, Lorenzo Nicolodi, Franz Pedit, Paul Peters, Boris Springborn, Yoshihiko Suyama, Ekkehard Tjaden, Konrad Voss; as well as my wife, Heike Jeromin.

Special thanks also go to F. Burstall for many helpful discussions, in particular on isothermic surfaces, and for his suggestion to investigate the "retraction form"; Y. Suyama for providing me with a new explicit example of a generic conformally flat hypersurface and for allowing me to include it; E. Tjaden for always having the time to discuss and optimize explicit examples, as well as for his help with computing/-er questions; and K. Voss for allowing me to include joint unpublished results on Willmore channel surfaces.

I am grateful for hospitality while (and for) working on this text at the Mathematisches Forschungsinstitut Oberwolfach and the Forschungsinstitut für Mathematik at ETH Zürich. This book is based on lecture notes of a course given by the author at TU Berlin (1999—2000) and on the author's Habilitation thesis, "Models in Möbius differential geometry," TU Berlin (2002).

The text was typeset using plain TeX, the sketches were prepared using `xfig`, and the surface graphics were produced using *Mathematica*.

[13] Of course, this does not reflect all relevant text passages; for example, Blaschke's book [29] is not cited in every relevant paragraph, while other references only provide a technical detail.

I also would like to express my gratitude to a referee of the manuscript who provided helpful suggestions, and to Roger Astley of Cambridge University Press, who sent it to this referee and with whom it was a pleasure to work anyway. For revising my English and for help during the preparation of the final manuscript, I would like to thank Elise Oranges.

Preliminaries

The Riemannian point of view

As already mentioned before, this book grew out of the lecture notes of a course given for graduate students who had previously taken a course in Riemannian geometry. This preliminary chapter is meant to be of a didactical (and historical) nature rather than to be a modern conceptual introduction to conformal geometry; a modern treatment[1)] of conformal geometry may be found in [254] or [50].

Thus, in this chapter, we will discuss basic notions and facts of conformal geometry from the point of view of Riemannian geometry: This is a point of view that most readers will be familiar with. The main goal will be to introduce the notion of the "conformal n-sphere" — this is the ambient space of the submanifolds that we are going to investigate — and to get some understanding of its geometry. In particular, we will discuss Riemannian spaces that are "conformally flat," that is, look locally like the conformal n-sphere.

In the second part of this chapter we will discuss the conformal geometry of submanifolds from the Riemannian point of view: We will deduce how the fundamental quantities of a submanifold change when the metric of the ambient space is conformally altered, and we will discuss various conformal and "Möbius invariants" that appear in the literature. Totally umbilical submanifolds will be treated in detail because they will serve as a main tool in our approach to Möbius geometry presented in the following chapters.

Much of the material in this chapter can be found in the two papers by R. Kulkarni [170] and J. Lafontaine [173]; see also the textbook [122].

The contents of this chapter are organized as follows:

Section P.1. The notions of conformal maps and conformal structures on manifolds are introduced and illustrated by various examples. The most important conformal map given in this section may be the stereographic projection. The metrics of constant curvature are discussed as representatives of the conformal structure given by the Euclidean metric. The term "conformal n-sphere" is defined.

Section P.2. In this section, the transformation formulas for the Levi-Civita connection and the curvature tensor under a conformal change of the Riemannian metric are derived.

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1) However, to appreciate the modern treatment of conformal geometry as a Cartan geometry, it should be rather helpful to be familiar with the classical model of Möbius differential geometry as it will be presented in Chapter 1 of the present book.

Section P.3. The Weyl and Schouten tensors are introduced via a decomposition of the Riemannian curvature tensor that conforms with the transformation behavior of the curvature tensor under conformal changes of the Riemannian metric.

Section P.4. The notion of conformal flatness is introduced and related to the existence of conformal coordinates. Then Lichtenstein's theorem on the conformal flatness of every surface (2-dimensional Riemannian manifold) is presented and two proofs are sketched. One of the proofs is based on the relation between conformal structures and complex structures on 2-dimensional manifolds.

Section P.5. The conformal flatness of higher dimensional Riemannian manifolds is discussed: The Weyl-Schouten theorem provides conditions for a Riemannian manifold of dimension $n \geq 3$ to be conformally flat. A proof of this theorem is given. As examples, spaces of constant sectional curvature and of 3-dimensional Riemannian product manifolds are given; the last example will become important when investigating conformally flat hypersurfaces in Chapter 2.

Section P.6. The geometric structures induced on a submanifold of a Riemannian manifold are introduced via the structure equations: the induced connection and the normal connection, the second fundamental form, and the Weingarten tensor field; and their transformation behavior under a conformal change of the ambient Riemannian structure is investigated. Some conformal and Möbius invariants are discussed. In particular, we arrive at Fubini's conformal fundamental forms for surfaces in the conformal 3-sphere, and we investigate Wang's "Möbius form," which turns out to be a Möbius invariant but not a conformal invariant; however, Rothe gave a conformally invariant formulation for Wang's Möbius form in the 2-dimensional case. Finally, it is shown that the notions of curvature direction and of umbilic are conformally invariant.

Section P.7. Using the conformal invariance of umbilics, we introduce hyperspheres as totally umbilic hypersurfaces that are maximal in an appropriate sense. As examples, we discuss the hyperspheres of the spaces of constant curvature — of particular interest may be the hyperspheres in hyperbolic space. Using Joachimsthal's theorem, spheres of higher codimension are then characterized in two equivalent ways. An ad-hoc definition is given for hyperspheres in the conformal 2-sphere S^2.

Section P.8. In this final section the notion of Möbius transformation is introduced, and Liouville's theorem on the relation between Möbius transformations and conformal transformations is formulated; a proof will be given later.

Remark. As indicated above, in this chapter the reader is expected to have some background in Riemannian geometry. On the other hand, most of the presented material will be familiar to the reader, so that this chapter may be omitted from a first reading; it may rather serve for reference. However, this chapter may be a good introduction when giving a course on the subject.

P.1 Conformal maps

First, we recall the notion of a conformal map between Riemannian manifolds and the notion of conformal equivalence of metrics:

P.1.1 Definition. *A map $f : (M, g) \to (\tilde{M}, \tilde{g})$ between Riemannian manifolds is called conformal*[2] *if the induced metric $f^*\tilde{g} = \tilde{g}(df, df) = e^{2u}g$ with some function $u : M \to \mathbb{R}$.*

Two metrics g and $\tilde{g}$ on $\tilde{M} = M$ are said to be conformally equivalent if the map $f = id$ is conformal.

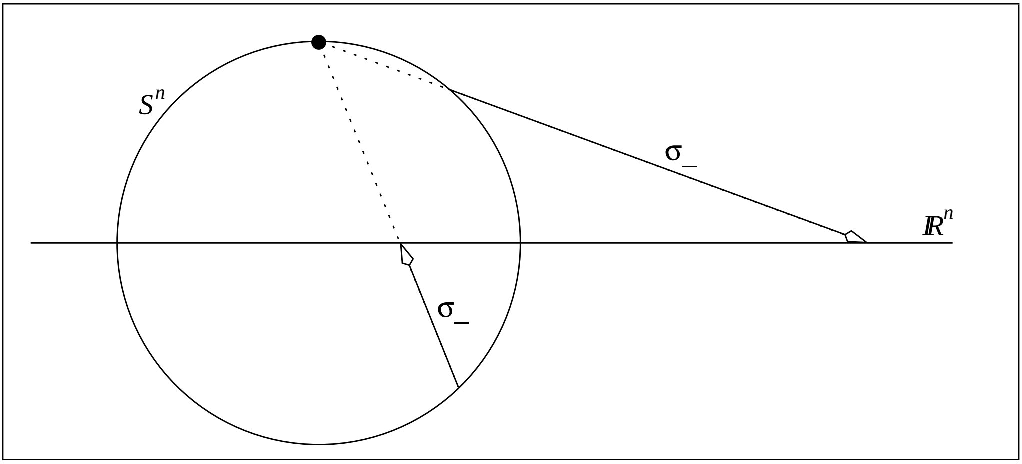

Fig. P.1. The stereographic projection

P.1.2 Note that a map $f : (M, g) \to (\tilde{M}, \tilde{g})$ is conformal iff it preserves angles iff it preserves orthogonality: Clearly, a conformal map preserves angles, and hence it preserves orthogonality. It remains to understand that f preserving orthogonality forces f to be conformal. For that purpose, let $(e_1, \ldots, e_n)$ denote an orthonormal basis of T_pM, with respect to g; then, $0 = f^*\tilde{g}_p(e_i, e_j)$ and $0 = f^*\tilde{g}_p(e_i + e_j, e_i - e_j) = f^*\tilde{g}_p(e_i, e_i) - f^*\tilde{g}_p(e_j, e_j)$ for $i \neq j$, showing that the $df(e_i)$ are orthogonal and have the same lengths. Hence f is conformal.

P.1.3 Examples. The stereographic projections $\sigma_\pm$, defined in terms of

[2] Note that, by definition, the conformal factor $e^{2u} > 0$ — allowing zeros of the conformal factor, the map is called "weakly conformal."

their inverses by

$$\sigma_\pm^{-1} : \mathbb{R}^n \to S^n \subset \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n, \quad p \mapsto \tfrac{1}{1+|p|^2}(\mp(1-|p|^2), 2p),$$

are conformal (cf., Figure P.1).

The Mercator map of the world (assuming the world is a 2-sphere), given by the parametrization $(u, v) \mapsto (\frac{1}{\cosh u}\cos v, \frac{1}{\cosh u}\sin v, \tanh u)$, is conformal;[3)] the Archimedes-Lambert projection (projection of the sphere to the cylinder along lines perpendicular to the axis) is area-preserving but not conformal. Note that the system of meridians and parallels of latitude still forms an orthogonal net (see [260]).

The metrics $g_k|_p = \frac{4dp^2}{(1+k|p|^2)^2}$, $k \geq k_0$ for some $k_0 \in \mathbb{R}$, are conformally equivalent on $\{p \in \mathbb{R}^n \,|\, 1 + k_0|p|^2 > 0\}$; in particular, the standard metrics of constant curvature[4)] $k \geq -\frac{1}{r^2}$ (cf., [122]) are conformally equivalent on the ball $B(r) := \{p \,|\, |p| < r\}$. The Poincaré ball model of hyperbolic space of curvature $k = -\frac{1}{r^2}$ is obtained as the only complete space in the family.

The metric $\frac{1}{p_0^2}(dp_0^2 + dp^2)$ on $(0, \infty) \times \mathbb{R}^n$ is clearly conformally equivalent to the standard Euclidean metric on that set as a subset of $\mathbb{R}^{n+1}$. This metric has constant sectional curvature -1 (cf., [162]) as one either computes directly or one concludes by defining an isometry onto the Poincaré ball model $(B(1), g_{-1})$ given above; this can be done by using a suitable "inversion" (we will learn about inversions later). This is the Poincaré half-space model of hyperbolic space.

P.1.4 Definition. *A conformal equivalence class of metrics on a manifold M is called a conformal structure on M.*

P.1.5 Clearly, "conformally equivalent" defines an equivalence relation for Riemannian metrics on a given manifold M. Thus, this definition makes sense.

Note that in [170] a slightly more general definition is given: There, a conformal structure is defined via locally defined metrics; however, any conformal structure in this wider sense contains a globally defined representative by a partition of the unity argument (see [170]).

One particular example of such a conformal structure on a specific manifold will become very important to us — as the ambient space of the submanifolds or hypersurfaces that we are going to examine:

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3) Note that the lines $v = const$ give unit speed geodesics in the Poincaré half-plane model (see the next example); the sphere is then isometrically parametrized as a surface in the space $S^1 \times H^2$, which is conformally equivalent to $\mathbb{R}^3 \setminus \{(0, 0, t)\}$ equipped with the standard Euclidean metric.

4) We will see later how to describe these models via "generalized stereographic projections."

P.1.6 Definition. *The n-sphere S^n equipped with its standard conformal structure will be called the conformal n-sphere.*

P.2 Transformation formulas

Our next goal is to give the transformation formulas for the Levi-Civita connection and the curvature tensor of a Riemannian manifold under conformal changes of the metric.

P.2.1 Lemma. *If $\tilde{g} = e^{2u}g$, then $\tilde{\nabla} = \nabla + B$ with the symmetric $(2,1)$-tensor field $B(x,y) := du(x)y + du(y)x - g(x,y)\operatorname{grad} u$.*

P.2.2 *Proof.* Obviously, $\tilde{\nabla} := \nabla + B$ defines a torsion-free connection as B is symmetric. Since

$$(\tilde{\nabla}_x \tilde{g})(y,z) = e^{2u}[2du(x)g(y,z) - g(B(x,y),z) - g(y,B(x,z))] = 0,$$

this is the Levi-Civita connection for $\tilde{g}$. ◁

The next definition is of a rather technical nature but allows us to write the transformation formulas for the curvature tensor in a much more condensed form (cf., [173]):

P.2.3 Definition. *Let $b_1, b_2 : V \times V \to \mathbb{R}$ be two symmetric bilinear forms on a vector space. The Kulkarni-Nomizu product of b_1 and b_2 is defined as*

$$(b_1 \wedge b_2)(x,y,z,w) := \begin{vmatrix} b_1(x,z) & b_1(x,w) \\ b_2(y,z) & b_2(y,w) \end{vmatrix} + \begin{vmatrix} b_2(x,z) & b_2(x,w) \\ b_1(y,z) & b_1(y,w) \end{vmatrix}.$$

P.2.4 Note that the Kulkarni-Nomizu product is bilinear (thus it qualifies as a "product") and symmetric, $b_1 \wedge b_2 = b_2 \wedge b_1$, and that it satisfies the same (algebraic) identities as a curvature tensor:

1. $b_1 \wedge b_2(x,y,w,z) = b_1 \wedge b_2(y,x,z,w) = -b_1 \wedge b_2(x,y,z,w)$,
2. $b_1 \wedge b_2(x,y,z,w) = b_1 \wedge b_2(z,w,x,y)$,
3. $b_1 \wedge b_2(x,y,z,w) + b_1 \wedge b_2(y,z,x,w) + b_1 \wedge b_2(z,x,y,w) = 0$.

These statements are proven by straightforward computation.

With this, the transformation formula for the curvature tensor reads

P.2.5 Lemma. *If $\tilde{g} = e^{2u}g$, then the $(4,0)$-curvature tensor of $\tilde{g}$ is given by $\tilde{r} = e^{2u}(r - b_u \wedge g)$, with[5] the symmetric bilinear form*

$$b_u(x,y) := \operatorname{hess} u(x,y) - du(x)du(y) + \tfrac{1}{2}g(\operatorname{grad} u, \operatorname{grad} u)g(x,y).$$

[5] We use the following sign convention: $R(x,y)z = -[\nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z]$.
Also note that, $\tilde{b}_{-u} = -b_u$ so that changing the metric conformally and then changing it back has no effect on the curvature tensor — as it should be.

P.2.6 *Proof.* Write $B = \tilde{\nabla} - \nabla$; then,

$$\begin{aligned}&\tilde{R}(x,y)z - R(x,y)z \\ &\quad = -(\nabla_x B)(y,z) + (\nabla_y B)(x,z) - B(x, B(y,z)) + B(y, B(x,z)).\end{aligned}$$

With $B(y,z) = du(y)z + du(z)y - g(y,z)\mathrm{grad}\, u$,

$$\begin{aligned}&g((\nabla_x B)(y,z), w) \\ &\quad = \mathrm{hess}\, u(x,y) g(z,w) + \mathrm{hess}\, u(x,z) g(y,w) - \mathrm{hess}\, u(x,w) g(y,z),\end{aligned}$$

$$\begin{aligned}&g(B(x, B(y,z)), w) \\ &\quad = \ [du(x)du(y)g(z,w) - g(x,y)du(z)du(w)] \\ &\qquad + [du(x)du(z)g(y,w) + 2g(x,w)du(y)du(z) - g(x,z)du(y)du(w)] \\ &\qquad - g(x,w)g(y,z)g(\mathrm{grad}\, u, \mathrm{grad}\, u).\end{aligned}$$

Adding the respective terms up, the claim follows. ◁

P.2.7 Corollary. *For the sectional curvatures, a conformal change of the metric* $g \to \tilde{g} = e^{2u} g$ *yields*[6] $\tilde{K}(x \wedge y) = e^{-2u}(K(x \wedge y) - \mathrm{tr}\, b_u|_{x\wedge y})$.

If $\dim M = 2$, *then* $\tilde{K} = e^{-2u}(K - \Delta u)$.

P.2.8 *Proof.* With an orthonormal basis (e_1, e_2) of $x \wedge y$ with respect to g, we have

$$\begin{aligned}\tilde{K}(x \wedge y) &= \tfrac{\tilde{r}(e_1,e_2,e_1,e_2)}{\tilde{g}(e_1,e_1)\tilde{g}(e_2,e_2)} \\ &= e^{-2u}(r(e_1,e_2,e_1,e_2) - b_u(e_1,e_1) - b_u(e_2,e_2)) \\ &= e^{-2u}(K(x \wedge y) - \mathrm{tr}\, b_u|_{x\wedge y}).\end{aligned}$$

If $\dim M = 2$, then $\mathrm{tr}\, b_u = \mathrm{tr}\,\mathrm{hess}\, u = \Delta u$. ◁

P.3 The Weyl and Schouten tensors

From the transformation formula for the $(4,0)$-curvature tensor we see that a conformal change of metric effects only the "trace part" of the curvature tensor, while the trace-free part is just scaled. This suggests a decomposition of the curvature tensor into a "trace part" and a "trace-free part" that is adapted to conformal geometry (cf., [122] or [173]).

P.3.1 Definition. *Let* (M, g) *be an* n*-dimensional Riemannian manifold. Then, the Weyl and Schouten tensors are defined as*

$$\begin{aligned} s &:= \tfrac{1}{n-2}\left(ric - \tfrac{scal}{2(n-1)} g\right) \qquad &\text{(Schouten tensor)}, \\ w &:= r - s \wedge g \qquad &\text{(Weyl tensor)}.\end{aligned}$$

[6] For the moment, "$x \wedge y$" just denotes the 2-plane spanned by the two vectors x and y.

P.3.2 The Weyl tensor is "trace-free": With an orthonormal basis e_i at p, we have $\sum_i w(x, e_i, y, e_i) = ric(x, y) - (n-2)s(x, y) - \mathrm{tr}\, s \cdot g(x, y) = 0$ indeed. On the other hand, if $r = w + s \wedge g$ with a "trace-free" tensor w, then

$$\begin{array}{rcl} ric & = & (n-2)s + \mathrm{tr}\, s \cdot g \\ scal & = & 2(n-1)\mathrm{tr}\, s, \end{array}$$

which implies that s is the Schouten tensor and w is the Weyl tensor (cf., [122]).

The transformation behavior of the curvature tensor is now easily described by that of the Weyl and Schouten tensors:

P.3.3 Lemma. *Under a conformal change $g \to \tilde{g} = e^{2u}g$ of the metric, the Weyl and Schouten tensors transform as follows:*

$$\begin{array}{rcccl} w & \to & \tilde{w} & = & e^{2u}w \ , \\ s & \to & \tilde{s} & = & s - b_u. \end{array}$$

In particular, the $(3,1)$-Weyl tensor W, $w = g(W.,.)$, is invariant under conformal changes of the metric; therefore, it is also called the "conformal curvature tensor."

P.4 Conformal flatness

As a first application of our notions, we want to discuss conformal flatness of a Riemannian manifold and give, in the following section, a criterion for conformal flatness in terms of the Weyl and Schouten tensors.

P.4.1 Definition. *A Riemannian manifold (M, g) is called conformally flat if, for any point $p \in M$, there is a neighbourhood U of p and some function $u : U \to \mathbb{R}$ so that the (local) metric $\tilde{g} = e^{2u}g$ is flat on U.*

P.4.2 Note that some authors define conformal flatness globally; that is, the conformal structure associated with the given metric is required to contain a (global) flat representative.

P.4.3 Lemma. *A manifold (M, g) is conformally flat if and only if, around each point $p \in M$, there exist conformal coordinates; that is, there is a coordinate map $x : M \supset U \to \mathbb{R}^n$ and a function $u : U \to \mathbb{R}$ such that the metric $g|_U = e^{2u} \sum_{i=1}^n dx_i^2$.*

P.4.4 *Proof.* Obviously, a metric of the form $g = e^{2u} \sum_{i=1}^n dx_i^2$ is conformally flat; on the other hand, to a flat metric $e^{-2u}g$, there always exist local coordinates x such that $e^{-2u}g = \sum_{i=1}^n dx_i^2$ (any flat manifold is locally isometric to $\mathbb{R}^n$, via the exponential map; cf., [122]). ◁

P.4.5 Any 1-dimensional Riemannian manifold (M^1, g) is conformally flat (actually: flat — there exist arc length parameters).

P.4.6 Theorem (Lichtenstein [183]). *Any 2-dimensional Riemannian manifold (M^2, g) is conformally flat.*[7)]

P.4.7 *Proof.* According to the transformation formula for the Gauss curvature from §P.2.7, we have to (locally) solve the partial differential equation $K = \Delta u$ for u. Then the (locally defined) metric $\tilde{g} := e^{2u} g$ will be flat.

P.4.8 Here is a sketch of a proof for the existence of local solutions:

On a 2-torus (T^2, g), the total curvature $\langle 1, K \rangle = \int_{T^2} 1 \cdot K \, dA = 0$ (Gauss-Bonnet theorem), that is, $K \in \ker^{\perp} \Delta = \{u : T^2 \to \mathbb{R} \,|\, u \equiv const\}$. By the Hodge decomposition theorem [298], $\ker^{\perp} \Delta = \mathrm{im}\Delta$ in $C^{\infty}(T^2, \mathbb{R})$; thus there exists a (global!) solution $u \in C^{\infty}(T^2)$ of $\Delta u = K$.

On an arbitrary (M^2, g) choose coordinates $x : M \supset U \to \mathbb{R}^2$ around a point $p \in M$ with $x(U) \subset (0,1)^2$ and introduce a metric $\hat{g}$ on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ such that $x^* \hat{g} = g$ on some neighborhood $\hat{U} \subset U$ of p (partition of unity). By the above argument, there exists a function $\hat{u} : T^2 \to \mathbb{R}$ such that $e^{2\hat{u}} \hat{g}$ is flat on T^2; since $x : (\hat{U}, g) \to (x(\tilde{U}), \hat{g})$ is an isometry, the metric $e^{2u} g|_{\hat{U}}$, with $u = \hat{u} \circ x$, is then flat on $\hat{U}$. ◁

P.4.9 The theorem of §P.4.6 establishes a relation between 2-dimensional manifolds equipped with a conformal structure and Riemann surfaces, that is, with 1-dimensional complex manifolds: Given isothermal (conformal) coordinates (x, y) around some $p \in M$ on (M^2, g), complex coordinates can be defined by $z := x + iy$. The transition functions of such complex coordinates are (as angle-preserving maps) either holomorphic or antiholomorphic; restricting to holomorphic (orientation preserving: M has to be orientable) transition functions, an atlas of complex coordinates is obtained. This identifies 2-dimensional (orientable) Riemannian manifolds as complex curves.

On the other hand, the pullback of the multiplication by i,

$$J_p : T_p M \to T_p M, \quad d_p z \circ J_p = i \cdot d_p z,$$

provides, if it can be extended to a global tensor field (that is, if M is orientable), 90° rotations (isometries with $J_p^2 = -id$) on each tangent space: an "almost complex structure." The above theorem states that any almost complex structure J on M (as it defines a conformal structure) comes from complex coordinate charts (that is, it is a "complex structure").

7) Compare also [182]. Concerning the realization of (compact) Riemann surfaces as submanifolds of Euclidean 3-space, see [124] and [125].

P.4.10 By using the concept of an almost complex structure, another (more constructive) proof for the existence of isothermal coordinates can be given (cf., [260]):

Suppose we are given[8)] a harmonic function $x_1 : M^2 \supset U \to \mathbb{R}$ on a simply connected neighborhood U around $p \in M$ such that $dx_1 \neq 0$; then we find[9)]

$$d(\star dx_1) := d(dx_1 \circ J) = dg(\operatorname{grad} x_1, J.) = -\Delta x_1\, dA = 0,$$

so that there is a function $x_2 : U \to \mathbb{R}$ with $dx_2 = \star dx_1$, the "conjugate harmonic function." Then, $(x_1, x_2) : U \to \mathbb{R}^2$ define isothermal coordinates since $\frac{\partial}{\partial x_2} = -J\frac{\partial}{\partial x_1}$ yields $0 = g(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ and $g(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}) = g(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1})$.

P.5 The Weyl-Schouten theorem

In higher dimensions, the Weyl-Schouten theorem gives a characterization of conformally flat Riemannian manifolds (cf., [81], [82], [300], [301], [252]):

P.5.1 Theorem (Weyl-Schouten). *A Riemannian manifold (M^n, g) of dimension $n \geq 3$ is conformally flat if and only if*
- *the Schouten tensor is a Codazzi tensor, $(\nabla_x s)(y, z) = (\nabla_y s)(x, z)$, in the case $n = 3$; and*
- *the Weyl tensor vanishes, $w \equiv 0$, in the case $n > 3$.*

P.5.2 *Proof.* (cf., [173]). By the transformation formula for the curvature tensor from §P.2.5, (M, g) is conformally flat if and only if $w = 0$ and there exists (locally) a function u with $s = b_u$.

We divide the proof into three steps:

P.5.3 Step 1. If $n = 3$, then $w = 0$, by algebra.

Let (e_1, e_2, e_3) be an orthonormal basis of some T_pM; then[10)]

$$r(e_i, e_j, e_i, e_j) = ric(e_i, e_i) + ric(e_j, e_j) - \tfrac{1}{2} scal = (s \wedge g)(e_i, e_j, e_i, e_j)$$

---

8) Of course, to obtain a complete proof, one would have to show that such nonconstant harmonic functions (locally) always exist.

9) Note that $\nabla J = 0$ because, for any vector field v, $g((\nabla J)v, v) = 0$ and $g((\nabla J)v, Jv) = 0$.

10) The computation is best done from right to left.
Another, more conceptual proof was pointed out by Konrad Voss: Let $Sym(V)$ denote the space of symmetric bilinear forms on a vector space V, and note that $Sym(\Lambda^2 V)$ is the space of algebraic curvature tensors on V if $\dim V = 3$ (the first Bianchi identity follows from the other symmetries in this case). Now, observe that

$$Sym(V) \ni s \mapsto r(s) = s \wedge g \in Sym(\Lambda^2 V) \quad \text{and} \quad Sym(\Lambda^2 V) \ni r \mapsto s(r) \in Sym(V)$$

are linear maps. Since $Sym(V) \ni s \mapsto s(r(s)) = s \in Sym(V)$ is the identity, the linear map $s \mapsto r(s)$ injects and hence is an isomorphism since $\dim Sym(V) = \dim Sym(\Lambda^2 V)$ in the case $\dim V = 3$.