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Contents

Metric $S^n \subset \mathbb{R}^{n+1}$	**Projective** $S^n \subset \mathbb{R}P^{n+1}$	**Minkowski** $L^{n+1} \subset \mathbb{R}^{n+2}_1$
point: $p \in S^n$	point: $p \in S^n \subset \mathbb{R}P^{n+1}$	lightlike vector $p \in \mathbb{R}^{n+2}_1,\ \langle p, p\rangle = 0$
sphere: $S \subset S^n$	point in "outer space": $S \in \mathbb{R}P^{n+1}_O$	spacelike vector: $S \in \mathbb{R}^{n+2}_1\ \langle S, S\rangle > 0$
incidence: $p \in S$	polarity: $S \in \mathrm{pol}[p] \cong T_pS^n$	orthogonality: $\langle p, S\rangle = 0$
orthogonal intersection: $S_1 \cap_\perp S_2$	polarity: $S_i \in \mathrm{pol}[S_j]$	orthogonality: $\langle S_1, S_2\rangle = 0$
intersection angle: $S_1 \cap_\alpha S_2$	—	scalar product: $\cos^2\alpha = \frac{\langle S_1,S_2\rangle^2}{\lvert S_1\rvert^2\lvert S_2\rvert^2}$
elliptic sphere pencil: $\{S \mid S_1 \cap S_2 \subset S\}$	line not meeting S^n: $\mathrm{inc}[S_1, S_2]$	spacelike 2-plane: $S_1 \wedge S_2, \langle S_1, S_2\rangle = 0$
parabolic sphere pencil: $\{S \mid p \in S, T_pS = T_pS_0\}$	line touching S^n: $\mathrm{inc}[p, S_0]$	degenerate 2-plane: $p \wedge S_0, \langle p, S_0\rangle = 0$
hyperbolic sphere pencil: $\{S \mid p_1, p_2 \in \tilde{S} \Rightarrow S \perp \tilde{S}\}$	line intersecting S^n: $\mathrm{inc}[p_1, p_2]$	Minkowski 2-plane: $p_1 \wedge p_2, p_1 \neq p_2$
elliptic sphere complex: $\{S \subset S^n \mid S \cap_\perp \mathcal{K} \cong S_\infty\}$	hyperplane intersecting S^n: $\mathrm{pol}[\mathcal{K}]$	Minkowski hyperplane: $\{\mathcal{K}\}^\perp, \langle\mathcal{K}, \mathcal{K}\rangle > 0$
parabolic sphere complex: $\{S \mid S \ni \mathcal{K} \cong p_\infty\}$	hyperplane touching S^n: $\mathrm{pol}[\mathcal{K}] \cong T_{p_\infty}S^n$	degenerate hyperplane: $\{\mathcal{K}\}^\perp, \langle\mathcal{K}, \mathcal{K}\rangle = 0$
hyperbolic sphere complex: $\{S \mid S^\perp \ni \mathcal{K} \cong 0 \in \mathbb{R}^{n+1}\}$	hyperplane not meeting S^n: $\mathrm{pol}[\mathcal{K}] \cong \mathbb{R}P^{n+1} \setminus \mathbb{R}^{n+1}$	spacelike hyperplane: $\{\mathcal{K}\}^\perp, \langle\mathcal{K}, \mathcal{K}\rangle < 0$
m-sphere: $S : p_i \in S, \dim S = m$	$(m+1)$-plane intersecting S^n: $\mathrm{inc}[p_1, \ldots, p_{m+2}]$	Minkowski $(m+2)$-space: $p_1 \wedge \ldots \wedge p_{m+2}$
$(n-m)$-sphere: $S_1 \cap \ldots \cap S_m$	$(m-1)$-plane in "outer space": $\mathrm{inc}[S_1, \ldots, S_m]$	spacelike m-space: $S_1 \wedge \ldots \wedge S_m$
inversion: $p \mapsto \frac{\sin^2(\varrho)p - 2(pm - \cos\varrho)m}{1 - 2pm\cos\varrho + \cos^2\varrho}$	polar reflection: –	reflection: $p \mapsto p - 2\frac{\langle p,S\rangle}{\langle S,S\rangle}S$
Möbius transformation: $\mu \in M\ddot{o}b(n)$	projective transformation: $\mu \in PGl(n+2), \mu(S^n) = S^n$	Lorentz transformation: $\mu \in O_1(n+2)$

Fig. T.1. The classical model of Möbius geometry

Metric $S^n \subset \mathbb{R}^{n+1}$	**Projective** $S^n \subset \mathbb{R}P^{n+1}$	**Minkowski** $L^{n+1} \subset \mathbb{R}^{n+2}_1$
space of const. curvature: $S^n_\kappa, \mathbb{R}^3, H^n_\kappa \dot\cup H^n_\kappa = M^n_\kappa$	... the base-mf becomes... $S^n \setminus \partial_\infty M^n_\kappa$	quadric: $Q^n_\kappa \subset L^{n+1}$
sphere complex: $\{S \subset M^n_\kappa \mid I\!I \equiv 0\}$	hyperplane: $\mathrm{pol}[\mathcal{K}]$	hyperplane: $\{\mathcal{K}\}^\perp$
immersion: $f : M^m \to S^n$	proj. immersion: $\forall p, v : f(p) \neq \partial_v f(p)$	spacelike immersion: $\langle df, df \rangle > 0$
sphere congruence: $M^m \ni p \mapsto S(p) \subset S^n$	sphere congruence: $S : M^m \to \mathbb{R}P^{n+1}_O$	sphere congruence (locally): $S : M^m \to S^{n+1}_1$
f envelopes S: $f(p) \in S(p)$, $d_p f(T_p M) \subset T_{f(p)} S(p)$	f envelopes S: $T_{f(p)} f(M) \subset \mathrm{pol}[S(p)]$	strip (f, S): $f(p), d_p f(T_p M) \perp S(p)$
Möbius frame: $(S_1, \ldots, S_{n-1}, S, f, \hat f)$	—	pseudo orthonormal frame: $F : M^m \to O_1(n+2)$

Fig. T.2. The classical model of Möbius geometry, differential geometric terms

Conformal $\mathbb{H}P^1$	**Homogeneous** $\mathbb{H}^2$	**Minkowski** $L^{n+1} \subset \mathfrak{H}(\mathbb{H}^2)$
point: $p = v\mathbb{H} \in S^4$	point: $v \in \mathbb{H}^2$	isotropic form: $S_p \in \mathfrak{H}(\mathbb{H}^2),\ \det S_p = 0$
hypersphere: $S \subset S^4$	—	spacelike form: $S \in \mathfrak{H}(\mathbb{H}^2),\ \det S < 0$
2-sphere: $\mathcal{S} \subset S^4$	involution: $\mathcal{S} \in \mathfrak{S}(\mathbb{H}^2)$	elliptic sphere pencil: $S, \mathcal{J}S \in \mathfrak{H}(\mathbb{H}^2)$
incidence: $p = v\mathbb{H} \in S$	isotropy: $S(v,v) = 0$	orthogonality: $\langle S_p, S\rangle = 0$
incidence: $p \in \mathcal{S}$	eigendirection: $\mathcal{S}v \parallel v$	orthogonality: $S_p \perp S, \mathcal{J}S$
intersection angle: $S_1 \cap_\alpha S_2$	—	scalar product: $\cos^2\alpha = \frac{\{S_1,S_2\}^2}{4S_1^2 S_2^2}$
f envelopes S: $f(p) \in S(p)$, $d_pf(T_pM) \subset T_{f(p)}S(p)$	f envelopes S: $S(f,f) = 0$ $S(f,df) + S(df,f) \equiv 0$	strip (S_f, S): $S_f(p), d_pS_f(T_pM) \perp S(p)$
f envelopes $\mathcal{S}$: $f(p) \in \mathcal{S}(p)$, $d_pf(T_pM) \subset T_{f(p)}\mathcal{S}(p)$	f envelopes $\mathcal{S}$: $\mathcal{S}f \parallel f$ $d\mathcal{S} \cdot f \parallel f$	—
Möbius transformation: $\mu \in M\ddot{o}b(4)$	fractional linear: $\mu \in Sl(2,\mathbb{H}),\ v \mapsto \mu v$	Lorentz transformation: $\mu \in Sl(2,\mathbb{H}),\ S \mapsto \mu S$
stereographic projection: $v \mapsto (\nu_0 v)(\nu_\infty v)^{-1}$	affine coordinates: $v = v_0 + v_\infty \mathfrak{p}$	—
point pair map: $(f,\hat{f}) : M \to \mathfrak{P}$	Möbius frame: $F : M \to Sl(2,\mathbb{H})$	Möbius frame: $(S_1,\ldots,S_4,S_f,S_{\hat{f}})$
cross-ratio: $[p_1;p_2;p_3;p_4]$	cross-ratio: $\nu_1 v_2 \frac{1}{\nu_3 v_2} \nu_1 v_4 \frac{1}{\nu_3 v_4}$	—

Fig. T.3. A quaternionic model of Möbius geometry

Conformal S^n	**Projective** $S^n \subset \mathbb{R}P^{n+1}$	**Clifford Algebra** $L^{n+1} \subset \Lambda^1\mathbb{R}^{n+2}_1$
point: $p \in S^n$	point: $p \in S^n$	isotropic vector $p \in \Lambda^1\mathbb{R}^{n+2}_1,\ p^2 = 0$
hypersphere: $s \subset S^n$	point in "outer space": $s \in \mathbb{R}P^{n+1}_O$	spacelike vector: $s \in \Lambda^1\mathbb{R}^{n+2}_1,\ s^2 < 0$
incidence: $p \in s$	polarity: $s \in \mathrm{pol}[p] \cong T_pS^n$	orthogonality: $\{p, s\} = 0$
orthogonal intersection: $s_1 \cap_\perp s_2$	polarity: $s_i \in \mathrm{pol}[s_j]$	orthogonality: $\{s_1, s_2\} = 0$
intersection angle: $s_1 \cap_\alpha s_2$	—	scalar product: $\cos^2\alpha = \frac{\{s_1,s_2\}^2}{4s_1^2s_2^2}$
k-sphere: $\mathfrak{s} : p_1, \dots, p_{k+2} \in \mathfrak{s}$	plane intersecting S^n: $\mathrm{inc}[p_1, \dots, p_{k+2}]$	timelike pure $(k+2)$-vector: $p_1 \wedge \dots \wedge p_{k+2} \in \Lambda^{k+2}\mathbb{R}^{n+2}_1$
k-sphere: $s_1 \cap \dots \cap s_{n-k}$	plane in "outer space": $\mathrm{inc}[s_1, \dots, s_{n-k}]$	spacelike pure $(n-k)$-vector: $s_1 \wedge \dots \wedge s_{n-k} \in \Lambda^{n-k}\mathbb{R}^{n+2}_1$
incidence: $p \in \mathfrak{s}$	polarity: $p \in \mathrm{pol}[\mathfrak{s}]$	vanishing of lower grade: $\mathfrak{s}p \in \Lambda^{n-k+1}\mathbb{R}^{n+2}_1$
f envelopes $\mathfrak{s}$: $f(p) \in \mathfrak{s}(p)$, $d_pf(T_pM) \subset T_{f(p)}\mathfrak{s}(p)$	f envelopes $\mathfrak{s}$: $T_{f(p)}f(M) \subset \mathrm{pol}[\mathfrak{s}(p)]$	no lower grades: $\mathfrak{s}f, \mathfrak{s}df \mapsto \Lambda^{n-k+1}\mathbb{R}^{n+2}_1$
inversion	polar reflection	reflection: $p \mapsto \frac{1}{\lvert s\rvert^2}sps$
Möbius transformation: $\mathfrak{z} \in M\ddot{o}b(n)$	projective transformation: $\mathfrak{z}$, $\mathfrak{z}(S^n) = S^n$	spinor: $\mathfrak{z} \in Spin_1(n+2)$
cross-ratio: $[p_1; p_2; p_3; p_4]$	—	cross-ratio: $\frac{p_1p_2p_3p_4 + p_4p_3p_2p_1}{(p_1p_4+p_4p_1)(p_2p_3+p_3p_2)}$

Fig. T.4. A Clifford algebra model of Möbius geometry

Conformal S^n	**Affine** $S^n \cong \mathbb{R}^n \cup \{\infty\}$	**Clifford Algebra** $L^{n+1} \subset \Lambda^1 \mathbb{R}^{n+2}_1$
point: $\mathcal{V} \in S^n$	"vector": $\binom{v}{1}$, $v \in \mathbb{R}^n$, or $\binom{1}{0}$	isotropic vector $\begin{pmatrix} v & -v^2 \\ 1 & -v \end{pmatrix}, \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \Lambda^1 \mathbb{R}^{n+2}_1$
hypersphere: $\mathcal{S} \subset S^n$	hypersphere/-plane: $\mathcal{S} \subset \mathbb{R}^n$	spacelike vector/Möbius involution: $\begin{pmatrix} m & -m^2-r^2 \\ 1 & -m \end{pmatrix}, \begin{pmatrix} n & 2d \\ 0 & -n \end{pmatrix} \in \Lambda^1 \mathbb{R}^{n+2}_1$
incidence: $\mathcal{V} \in \mathcal{S}$	fixed point: $\mathcal{S}\binom{v}{1} = \binom{v}{1} a$	orthogonality: $\{\mathcal{V}, \mathcal{S}\} = 0$
orthogonal intersection: $\mathcal{S}_1 \cap_\perp \mathcal{S}_2$	—	orthogonality: $\{\mathcal{S}_1, \mathcal{S}_2\} = 0$
intersection angle: $\mathcal{S}_1 \cap_\alpha \mathcal{S}_2$	—	scalar product: $\cos^2 \alpha = \frac{\{\mathcal{S}_1, \mathcal{S}_2\}^2}{4\mathcal{S}_1^2 \mathcal{S}_2^2}$
k-sphere: $\int : \mathcal{V}_1, \dots, \mathcal{V}_{k+2} \in \int$	—	timelike pure $(k+2)$-vector: $\mathcal{V}_1 \wedge \dots \wedge \mathcal{V}_{k+2} \in \Lambda^{k+2} \mathbb{R}^{n+2}_1$
k-sphere: $\mathcal{S}_1 \cap \dots \cap \mathcal{S}_{n-k}$	—	spacelike pure $(n-k)$-vector: $\mathcal{S}_1 \wedge \dots \wedge \mathcal{S}_{n-k} \in \Lambda^{n-k} \mathbb{R}^{n+2}_1$
incidence: $v \in \mathcal{S}$	fixed point: $\mathcal{S}\binom{v}{1} = \binom{v}{1} a$	fixed point: $r(\mathcal{S})\mathcal{V} \parallel \mathcal{V}$
inversion	inversion: $v \mapsto m - r^2(v-m)^{-1}$	reflection: $\mathcal{V} \mapsto \frac{1}{\lvert\mathcal{S}\rvert^2} \mathcal{S}\mathcal{V}\mathcal{S}$
Möbius transformation: $\mu \in M\ddot{o}b(n)$	fractional linear: $v \mapsto (av+b)(cv+d)^{-1}$	spinor: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Pin(\mathbb{R}^{n+2}_1)$
point pair map: $(f_\infty, f_0) : M \to \mathfrak{P}$	point pair map $(f_\infty, f_0), f_i : M \to \mathbb{R}^n$	Möbius frame: $\begin{pmatrix} f_\infty & f_0 \\ 1 & 1 \end{pmatrix} : M \to \Gamma(\mathbb{R}^{n+2}_1)$

Fig. T.5. A Clifford algebra model: Vahlen matrices