Chapter 1

Introduction

1.1 Solitons and soliton equations

Ever since the observation of the "great wave of translation" in water waves, by J. Scott Russell in 1834 [146, 147] while he rode on horseback near a narrow canal in Edinburgh, localized (nonoscillatory) solitary waves have been known to researchers studying wave dynamics. Despite Russell's detailed observations, it was many years before mathematicians formulated the relevant equation, now known as the Korteweg–de Vries (KdV) equation that governs those waves (cf. [38, 39, 112]). From the period 1895–1960, the study of water waves was essentially the only application in which solitary waves were found. However, in the 1960s it was discovered that the KdV equation is a relevant model in many other physical contexts, such as plasma physics, internal waves, lattice dynamics, and others. Critically, in their study of the Fermi–Pasta–Ulam lattice equation [75] Zabusky and Kruskal (1965) found that the KdV equation was the governing equation (cf. [189]). Moreover, in a wholly new discovery, Zabusky and Kruskal observed that the solitary waves of KdV are "elastic" in their interaction. That is, the solitary waves pass through one another and subsequently retain their characteristic form and velocity. Zabusky and Kruskal called these elastically interacting solitary waves *solitons*. The work of Gardner, Green, Kruskal, and Miura [82] showed how direct and inverse scattering techniques could be used to linearize the initial-value problem of KdV. The solitons also were shown to correspond to eigenvalues of the time-independent Schrödinger equation. The remarkable discovery of the soliton was the first in a chain of events that culminated in a mathematical theory of solitons in KdV. This work opened a rich vein of research that continues today, more than 35 years later. Further discussion of both the historical background and subsequent development of soliton theory can be found in [21].

Subsequent to the development of a mathematical theory of the solitons of KdV, further research revealed that solitons, with their distinctive elastic interactions, arise in numerous important physical systems. These systems are governed, respectively, by a diverse collection of evolution equations that are characterized mainly by the fact that they admit soliton solutions. For example, soliton solutions have been found in a number of nonlinear partial differential equations in $1+1$ dimensions (i.e., one space and one time dimension) and $2+1$ (i.e., two spatial dimensions and one time dimension). In addition, soliton solutions have been found in semi-discrete (discrete in space, continuous in time) and doubly discrete (discrete in space and time) nonlinear evolution equations and in nonlinear singular integro-differential equations, among others. A survey of some of these can be found in [6]. It should also be noted that there is a four-dimensional system, referred to as the self-dual Yang–Mills (SDYM) equations, that plays an important role in the study of soliton theory or integrable systems. Indeed, the SDYM equations can be viewed as a "master" integrable system from which virtually all other systems can be obtained as special reductions (cf. Atiyah and Ward [25]; Ward [181, 182, 183]; Belavin and Zakharov [29]; Mason et al. [124, 125]; Chakravarty et al. [50, 51]; Maszczyk et al. [126]; Ablowitz et al. [5]).

Researchers in physics and engineering have understood that stable localized solitary waves, even those that do not have the special property of elastic interaction, have many important applications, and their study has led to substantial research in specialized fields (e.g., nonlinear optics) all by itself. Nevertheless, the systems in which the solitary waves interact elastically, the "true" soliton systems, are important special cases. Moreover, there is an intrinsic richness in the mathematical theory of these soliton systems. Accordingly, the field of research associated with integrable systems has grown, developed, and expanded in many directions. One centrally important issue is the method of solution, sometimes referred to as the inverse scattering transform (IST), for these soliton equations. For a number of physically significant equations, the IST can be carried out in an explicit, effective, and illuminative manner. In particular, the IST is a fruitful approach, and it is the basis for the study of the nonlinear Schrödinger systems described in this book.

There are numerous books, review articles, and edited collections (see, for instance, [6, 21, 47, 65, 74, 140]) that delve widely and deeply into the theory of integrable systems. In this book, we give a detailed description of both the IST and the soliton solutions of integrable nonlinear Schrödinger systems, which are mathematically and physically important soliton equations. The collection of systems examined in this book comprises both continuous and semi-discrete systems of equations. As will be described in Section 1.4, these particular systems arise in the modeling of a wide array of physical wave phenomena. In this book, we present most of the known results for these nonlinear Schrödinger systems, as well as some new ones, in a comprehensive, unified framework built with the mathematical machinery of the inverse scattering transform.

1.2 The inverse scattering transform – Overview

The IST is a method that allows one to linearize a class of nonlinear evolution equations. In doing so one can obtain global information about the structure of the solution. In many respects, one can view the IST as a nonlinear version of the Fourier transform.

The solution of the initial-value problem of a nonlinear evolution equation by IST proceeds in three steps, as follows:

1. the forward problem – the transformation of the initial data from the original "physical" variables to the transformed "scattering" variables;
2. time-dependence – the evolution of the transformed data according to simple, explicitly solvable evolution equations;
3. the inverse problem – the recovery of the evolved solution in the original variables from the evolved solution in the transformed variables.

In fact, with the IST machinery one can do more than solve the initial-value problem; one also can construct special solutions of the evolution equation by positing an elementary solution in the transformed variables and then applying the inverse transformation to obtain the corresponding solution in the original variables. In general, the soliton and multisoliton solutions of soliton equations can be constructed in this way. In particular, in the subsequent chapters of this book, we explicitly construct the soliton solutions of four different nonlinear Schrödinger (NLS) systems. Moreover, one can in principle obtain the long-time asymptotics of solutions. In this book, we obtain formulas for the collision-induced phase shifts of solitons in NLS systems, including the polarization shift of solitons in the vector systems, from the asymptotics of the associated scattering data.

An essential prerequisite of the IST method is the association of the nonlinear evolution equation with a pair of linear problems, a linear eigenvalue problem and a second associated linear problem, such that the given evolution equation results as the compatibility condition between them. The pair of linear operators used to construct the associated linear problems is sometimes referred to as a "Lax pair," due to a formulation by Lax [115]. The solution of the nonlinear evolution equation appears as a coefficient in the associated linear eigenvalue problem. For example, in the work of Gardner et al., the solution of the KdV equation is associated with the potential in the linear Schrödinger equation. The eigenvalues and continuous spectrum of this linear eigenvalue problem constitute the transformed variables. The second associated linear problem determines the evolution of the transformed variables.

The associated eigenvalue problem introduces an intermediate stage in both the forward and inverse problems of the IST. In the forward problem, the first step is to construct eigenfunction solutions of the associated linear problem. These eigenfunctions depend on both the original spatial variables and the spectral parameter (eigenvalue). Second, with these eigenfunctions, one determines *scattering data* that are independent of the original spatial variables. In the inverse problem, the first step is the recovery of the eigenfunctions from the (evolved) scattering data. Finally, one recovers the solution in the original variables from these (evolved) eigenfunctions. As noted previously, the evolution of the scattering data is determined by the second associated linear operator and can be computed explicitly.

The properties of the eigenfunctions are key to the formulation of the inverse problem. In general, the solutions of the associated eigenvalue problem also satisfy linear integral equations. For the NLS systems discussed here (as well as other soliton equations in $1+1$ dimensions), by using such integral equations one can show that the eigenfunctions are sectionally analytic functions of the spectral parameter. By taking into account the analyticity properties of the eigenfunctions, one can formulate the inverse problem (in particular, the recovery of the eigenfunctions from the scattering data) as a generalized Riemann–Hilbert problem. The Riemann–Hilbert problem is then transformed into a system of linear algebraic–integral equations. Typically in the formulation of the IST, and in particular for the nonlinear Schrödinger systems considered in this book, the scattering data satisfy symmetry relations that are independent of the evolution. These symmetries in the scattering data are essential and must be taken into account in the solution of the inverse problem.

As explained previously, to apply the IST to a nonlinear evolution equation, one must first find a pair of linear operators that can be associated with the nonlinear equation. However, the general method of construction of the Lax pair for a given evolution equation remains on open problem. Nevertheless, for several equations of physical and mathematical interest, such a pair has been found and the IST developed.

In a major step forward, following the works of Gardner et al. [82] and Lax [115], in 1972 Zakharov and Shabat [192] showed that the method also could be applied to another physically significant nonlinear evolution equation, namely, the nonlinear Schrödinger equation (NLS). Subsequently, Manakov extended this approach to the solution of a pair of coupled NLS equations [120]. In fact, Manakov's work applies equally well to a system of N coupled NLS equations, a system that we refer to as vector NLS (VNLS). Using these ideas, Ablowitz, Kaup, Newell, and Segur developed a method to find a rather wide class of nonlinear evolution equations solvable by this technique [10]. In their work they named the technique the inverse scattering transform (IST). Later, Beals and Coifman analyzed the direct and inverse scattering associated with higher order systems of linear operators [28].

The IST has been extended to semi-discrete nonlinear evolution equations (discrete in space and continuous in time) as well as doubly discrete (discrete in both space and time) systems. Flaschka adapted the IST to solve the Toda lattice equation [79], and Manakov used a similar formulation to solve a nonlinear ladder network [121]. Subsequently, Ablowitz and Ladik developed a method to construct families of semi-discrete and doubly discrete nonlinear systems along with their respective linear operator pairs, as required for the solution of the nonlinear systems via the IST [11, 12] (see also [21]). Included in the formulation of Ablowitz and Ladik are an integrable semi-discretization of NLS (which we refer to as integrable discrete NLS, or IDNLS) as well as a doubly discrete integrable NLS. In Chapter 5 we will further extend the IST method to an integrable semi-discretization of the VNLS that was introduced in [18].

While the work mentioned in the preceding paragraphs consists of applications of the IST method to $1+1$–dimensional evolution equations, since the early 1980s significant progress has also been made in the extension of the IST approach to $2+1$–dimensional systems. For example, the Kadomtsev–Petviashvili equation [100], which is a $2+1$–dimensional generalization of the KdV equation, and the Davey–Stewartson equation [59], which is a natural $2+1$–dimensional integrable extension of the NLS equation, can be solved via the IST. However, the extension of the IST to such systems is beyond the scope of this book. A review of some developments in the application of the IST to $2+1$–dimensional systems can be found in [6].

It should also be noted that IST for periodic and other boundary conditions has been considered, but we will not discuss this here. Additional references can be found in the Bibliography.

1.3 Nonlinear Schrödinger systems

The scalar nonlinear Schrödinger (NLS) equation

$$iq_t = q_{xx} \pm 2\,|q|^2\,q \tag{1.3.1}$$

is a physically and mathematically significant nonlinear evolution equation. It results from the coupled pair of nonlinear evolution equations

$$iq_t = q_{xx} - 2rq^2 \tag{1.3.2a}$$

$$-ir_t = r_{xx} - 2qr^2 \tag{1.3.2b}$$

if we let $r = \mp q^*$.

The NLS equation (1.3.1) arises in a generic situation. It describes the evolution of small amplitude, slowly varying wave packets in nonlinear media [30]. Indeed, it has been derived in such diverse fields as deep water waves [190, 31]; plasma physics [191]; nonlinear optical fibers [91, 92]; magneto-static spin waves [194]; and so on. Mathematically, it attains broad significance because it is integrable by the IST [192], it admits soliton solutions, it has an infinite number of conserved quantities, and so on.

We also note that the form of the NLS equation (1.3.1) with a minus sign in front of the nonlinear term is sometimes referred to as the "defocusing" case. The defocusing NLS equation does not admit soliton solutions that vanish at infinity. However, it does admit soliton solutions that have a nontrivial background intensity (called dark solitons) [92, 193]. We will only discuss the IST for functions decaying sufficiently rapidly at infinity.

The vector nonlinear Schrödinger equation,

$$iq_t^{(1)} = q_{xx}^{(1)} + 2\left(|q^{(1)}|^2 + |q^{(2)}|^2\right) q^{(1)} \tag{1.3.3a}$$

$$iq_t^{(2)} = q_{xx}^{(2)} + 2\left(|q^{(1)}|^2 + |q^{(2)}|^2\right) q^{(2)}, \tag{1.3.3b}$$

arises, physically, under conditions similar to those described by the NLS when there are two wavetrains moving with nearly the same group velocities [144, 185]. Moreover, VNLS models physical systems in which the field has more than one component; for example, in optical fibers and waveguides, the propagating electric field has two components that are transverse to the direction of propagation. Manakov [120] first examined equation (1.3.3) as an asymptotic model for the propagation of the electric field in a wageguide. Subsequently, this system was derived as a key model for lightwave propagation in optical fibers (cf. [72], [122], [131], [179]).

In the literature, the system (1.3.3) is sometimes referred to as the coupled NLS equation. This system admits vector–soliton solutions, and the soliton collision is elastic. Moreover, the dynamics of soliton interactions can be explicitly computed [120]. (In [142] a different point of view is discussed.) Vector–soliton collisions are analyzed in Sections 4.3 (continuous) and 5.3 (discrete). In these sections, the elasticity of vector–soliton interactions and the order-dependence of these interactions are described in detail.

Both the VNLS equation (1.3.3) and its generalization,

$$i\mathbf{q}_t = \mathbf{q}_{xx} \pm 2\,\|\mathbf{q}\|^2\,\mathbf{q}, \tag{1.3.4}$$

where $\mathbf{q}$ is an N-component vector and $\|\cdot\|$ is the Euclidean norm, are integrable by the IST. In [120] only the case $N = 2$ is studied, but the extension to more components is straightforward. The N-component equation can be derived, with some additional conditions, as an asymptotic model of the interaction of N wavetrains in a weakly nonlinear, conservative medium (cf. [144]).

In optical fibers and waveguides, depending on the physics of the particular system, the propagation of the electromagnetic waves may be described by variations of equation (1.3.3). Note that the VNLS equation is the ideal (exactly integrable) case. For example, a model with physical significance is [131, 132, 150, 184, 187]

$$iq_t^{(1)} = q_{xx}^{(1)} + 2\left(|q^{(1)}|^2 + B|q^{(2)}|^2\right)q^{(1)} \tag{1.3.5a}$$

$$iq_t^{(2)} = q_{xx}^{(2)} + 2\left(B|q^{(1)}|^2 + |q^{(2)}|^2\right)q^{(2)}, \tag{1.3.5b}$$

which is equivalent to equation (1.3.3) when $B = 1$. However, based on the properties of equations (1.3.5), apparently it is not integrable when $B \neq 1$ (see the discussion in [18]).

The VNLS (1.3.4) has a natural matrix generalization in the system

$$i\mathbf{Q}_t = \mathbf{Q}_{xx} - 2\mathbf{QRQ} \tag{1.3.6a}$$

$$-i\mathbf{R}_t = \mathbf{R}_{xx} - 2\mathbf{RQR}, \tag{1.3.6b}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are $N \times M$ and $M \times N$ matrices, respectively. When $\mathbf{R} = \mp\mathbf{Q}^H$ (here and in the following, the superscript H denotes the Hermitian, i.e., conjugate transpose), the system (1.3.6a)–(1.3.6b) reduces to the single matrix equation

$$i\mathbf{Q}_t = \mathbf{Q}_{xx} \pm 2\mathbf{QQ}^H\mathbf{Q}, \tag{1.3.7}$$

which we refer to as matrix NLS or MNLS. The VNLS corresponds to the special case when $\mathbf{Q}$ is an N-component row vector and $\mathbf{R}$ is an N-component column vector, or vice versa. In particular, we obtain the system (1.3.3) when $M = 1$ and $N = 2$.

Both the NLS and the VNLS equations admit integrable discretizations that, besides being used as the basis for constructing numerical schemes for the continuous counterparts, also have physical applications as discrete systems (see, e.g., Aceves et al. [22, 23]; Braun and Kivshar [40]; Christodoulides and Joseph [54]; Claude et al. [55]; Darmanyan et al. [57, 58]; Davydov [60, 61, 62]; Eilbeck et al. [69]; Eisenberg et al. [70, 71]; Flach et al. [76, 77];

Its et al. [97]; Kenkre et al. [102, 103]; Kivshar, Kivshar, and Luther-Davies [106, 107]; Lederer et al. [116, 117]; Malomed and Weinstein [118]; Morandotti et al. [136, 137, 138]; Scott and Macneil [148]; Vakhnenko et al. [173, 174]).

A natural discretization of NLS (1.3.1) is the following:

$$i\frac{d}{dt}q_n = \frac{1}{h^2}\left(q_{n+1} - 2q_n + q_{n-1}\right) \pm \left|q_n\right|^2 \left(q_{n+1} + q_{n-1}\right), \tag{1.3.8}$$

which is referred to here as the integrable discrete NLS (IDNLS). It is a $O(h^2)$ finite-difference approximation of (1.3.1) that is integrable via the IST and has soliton solutions on the infinite lattice [11], [12]. We note that, if we change the nonlinear term in (1.3.8) to $2\left|q_n\right|^2 q_n$, the equation, which is often called the discrete NLS (DNLS) equation, is apparently no longer integrable, and it has been found that in certain circumstances chaotic dynamics results [19]. It should be remarked that the (apparently nonintegrable) DNLS equation arises in many important physical contexts (cf. [22], [23], [40], [54], [70], [71], [76], [104], [107], [116], and [136]–[138]). See also [17], [68], [105] for additional useful references.

Correspondingly, we will consider the discretization of the VNLS given by the following system:

$$i\frac{d}{dt}\mathbf{q}_n = \frac{1}{h^2}\left(\mathbf{q}_{n+1} - 2\mathbf{q}_n + \mathbf{q}_{n-1}\right) - \mathbf{r}_n \cdot \mathbf{q}_n \left(\mathbf{q}_{n+1} + \mathbf{q}_{n-1}\right) \tag{1.3.9a}$$

$$-i\frac{d}{dt}\mathbf{r}_n = \frac{1}{h^2}\left(\mathbf{r}_{n+1} - 2\mathbf{r}_n + \mathbf{r}_{n-1}\right) - \mathbf{r}_n \cdot \mathbf{q}_n \left(\mathbf{r}_{n+1} + \mathbf{r}_{n-1}\right), \tag{1.3.9b}$$

where $\mathbf{q}_n$ and $\mathbf{r}_n$ are N-component vectors and $\cdot$ is the inner product. Under the symmetry reduction $\mathbf{r}_n = \mp\mathbf{q}_n^*$ (here and in the following * indicates the complex conjugate), the system (1.3.9a)–(1.3.9b) reduces to the single equation

$$i\frac{d}{dt}\mathbf{q}_n = \frac{1}{h^2}\left(\mathbf{q}_{n+1} - 2\mathbf{q}_n + \mathbf{q}_{n-1}\right) \pm \left\|\mathbf{q}_n\right\|^2 \left(\mathbf{q}_{n+1} + \mathbf{q}_{n-1}\right), \tag{1.3.10}$$

which, for $\mathbf{q}_n = \mathbf{q}(nh)$ in the limit $h \to 0$, $nh = x$, gives the VNLS (1.3.4). In [18] it was shown that its solitary wave solutions interact elastically and that (1.3.10) admits multisoliton solutions. Thus the expectation was that the discrete vector NLS system (1.3.10) is indeed integrable. We refer to (1.3.10) as the integrable discrete vector NLS (IDVNLS).

An associated pair of linear operators (Lax pair) for the system (1.3.9a)–(1.3.9b) was constructed in [170]. In fact, the Lax pair for the vector system (1.3.10) is a reduction of a *matrix* generalization of the Lax associated with IDNLS. The matrix analog of the vector system (1.3.9) is given by

$$i\frac{d}{d\tau}\mathbf{Q}_n = \mathbf{Q}_{n+1} - 2\mathbf{Q}_n + \mathbf{A}\mathbf{Q}_n + \mathbf{Q}_n\mathbf{B} + \mathbf{Q}_{n-1} - \mathbf{Q}_{n+1}\mathbf{R}_n\mathbf{Q}_n - \mathbf{Q}_n\mathbf{R}_n\mathbf{Q}_{n-1} \tag{1.3.11a}$$

$$-i\frac{d}{d\tau}\mathbf{R}_n = \mathbf{R}_{n+1} - 2\mathbf{R}_n + \mathbf{B}\mathbf{R}_n + \mathbf{R}_n\mathbf{A} + \mathbf{R}_{n-1} - \mathbf{R}_{n+1}\mathbf{Q}_n\mathbf{R}_n - \mathbf{R}_n\mathbf{Q}_n\mathbf{R}_{n-1}, \tag{1.3.11b}$$

where $\mathbf{Q}_n$, $\mathbf{R}_n$ are $N \times M$ and $M \times N$ matrices, respectively, $\mathbf{A}$ is an $N \times N$ diagonal matrix, and $\mathbf{B}$ is an $M \times M$ diagonal matrix. $\mathbf{A}$ and $\mathbf{B}$ represent a gauge freedom in the definition of the integrable discrete MNLS (IDMNLS) that will be used in the following. In [83], [84] the IST for an eigenvalue problem that is equivalent to the scattering problem considered in [18], [19], [167], [170] had been formulated.

Note that the system (1.3.11a)–(1.3.11b) does not, in general, admit the reduction

$$\mathbf{R}_n = \mp\mathbf{Q}_n^H. \tag{1.3.12}$$

However, for $N = M$ one can restrict $\mathbf{R}_n$ and $\mathbf{Q}_n$ to be such that

$$\mathbf{R}_n\mathbf{Q}_n = \mathbf{Q}_n\mathbf{R}_n = \alpha_n\mathbf{I}_N, \tag{1.3.13}$$

where $\mathbf{I}_N$ is the identity $N \times N$ matrix and α_n is a scalar, and with this restriction $\mathbf{R}_n = \mp\mathbf{Q}_n^H$ is a consistent reduction of the system (1.3.11a)–(1.3.11b) that results in the single matrix equation

$$i\frac{d}{d\tau}\mathbf{Q}_n = \mathbf{Q}_{n+1} - 2\mathbf{Q}_n + \mathbf{A}\mathbf{Q}_n + \mathbf{Q}_n\mathbf{B} + \mathbf{Q}_{n-1} \mp \mathbf{Q}_n\mathbf{Q}_n^H\left(\mathbf{Q}_{n+1} + \mathbf{Q}_{n-1}\right). \tag{1.3.14}$$

Similarly, the IST for (1.3.14) follows the same lines as that for (1.3.11a)–(1.3.11b) with additional symmetry conditions imposed. The additional symmetry (1.3.13) (which has no analog in the continuous case) has essential consequences for the IST, which are discussed in detail in Section 5.2.2.

1.4 Physical applications

As indicated in the previous section, NLS systems have broad application in physical problems. In this section we briefly describe how certain NLS systems arise in nonlinear optics. We choose to discuss nonlinear optics because of its many scientific and technological applications. Here we will only sketch the key ideas behind the derivation of the NLS equations for some of the nonlinear optics applications. Interested readers will be able to find additional details and applications in the cited references. It should also be noted that this section can be read independently from the text describing the IST analysis.

We begin with a discussion of pulse propagation in optical fibers. Among the physical properties of optical fibers, nonlinearity and dispersion are serious sources of signal distortion. Signal loss in fibers was also a major limitation;

however, in the 1980s this was largely overcome due to the development of all-optical amplifiers (cf. [24]).

Dispersion originates from the frequency dependence of the refractive index of the fiber and leads to frequency-dependence of the group velocity; this is usually called group velocity dispersion or simply GVD. Due to GVD, different spectral components of an optical pulse propagate at different group velocities and thus arrive at different times. This leads to pulse broadening, resulting in signal distortion.

Fiber nonlinearity is due to the so-called Kerr effect, where the refractive index depends on the intensity of the optical pulse. In the presence of GVD and Kerr nonlinearity, the refractive index is expressed as

$$n(\omega, E) = n_0(\omega) + n_2|E|^2, \qquad (1.4.15)$$

where ω and E represent the frequency and electric field of the lightwave, respectively; $n_0(\omega)$ is the frequency-dependent linear refractive index; and the constant n_2, referred to as the Kerr coefficient, has a value of approximately10^{-22} m^2/W. Even though fiber nonlinearity is small, the nonlinear effects accumulate over long distances and can have a significant impact due to the high intensity of the lightwave over the small fiber cross section. By itself, the Kerr nonlinearity produces an intensity-dependent phase shift that results in spectral broadening during propagation.

In the usual transmission process with lightwaves, the electric field is modulated into a slowly varying amplitude of a carrier wave. Concretely, a modulated electromagnetic lightwave is written as

$$E(z, t) = \mathcal{E}(z, t)e^{i(k_0 z - \omega_0 t)} + c.c., \qquad (1.4.16)$$

where c.c. denotes complex conjugation, z the distance along the fiber, t the time, $k_0 = k_0(\omega_0)$ the wavenumber, ω_0 the frequency, and $\mathcal{E}(z, t)$ the envelope of the electromagnetic field.

Hasegawa and Tappert [91] first derived the NLS equation in the context of fiber optics. Detailed derivations can be found in texts (cf. Hasegawa and Kodama [90] and references therein). A simplified derivation is conveniently obtained from the nonlinear dispersion relation:

$$k(\omega, E) = \frac{\omega}{c}\left(n_0(\omega) + n_2|E|^2\right), \qquad (1.4.17)$$

where c denotes the speed of light.

A Taylor series expansion of $k(\omega, E)$ around the carrier frequency $\omega = \omega_0$ yields

$$k - k_0 = k'(\omega_0)(\omega - \omega_0) + \frac{k''(\omega_0)}{2}(\omega - \omega_0)^2 + \frac{\omega_0 n_2}{c}|E|^2, \qquad (1.4.18)$$