Part I

Fundamentals and Techniques of Complex Function Theory

The first portion of this text aims to introduce the reader to the basic notions and methods in complex analysis. The standard properties of real numbers and the calculus of real variables are assumed. When necessary, a rigorous axiomatic development will be sacrificed in place of a logical development based upon suitable assumptions. This will allow us to concentrate more on examples and applications that our experience has demonstrated to be useful for the student first introduced to the subject. However, the important theorems are stated and proved.

1

Complex Numbers and Elementary Functions

This chapter introduces complex numbers, elementary complex functions, and their basic properties. It will be seen that complex numbers have a simple two-dimensional character that submits to a straightforward geometric description. While many results of real variable calculus carry over, some very important novel and useful notions appear in the calculus of complex functions. Applications to differential equations are briefly discussed in this chapter.

1.1 Complex Numbers and Their Properties

In this text we use Euler's notation for the imaginary unit number:

$$i^2 = -1 \tag{1.1.1}$$

A complex number is an expression of the form

$$z = x + iy \tag{1.1.2}$$

Here x is the real part of z, $\text{Re}(z)$; and y is the imaginary part of z, $\text{Im}(z)$. If $y = 0$, we say that z is real; and if $x = 0$, we say that z is pure imaginary. We often denote z, an element of the complex numbers as $z \in \mathbb{C}$; where x, an element of the real numbers is denoted by $x \in \mathbb{R}$. Geometrically, we represent Eq. (1.1.2) in a two-dimensional coordinate system called the **complex plane** (see Figure 1.1.1).

The real numbers lie on the horizontal axis and pure imaginary numbers on the vertical axis. The analogy with two-dimensional vectors is immediate. A complex number $z = x + iy$ can be interpreted as a two-dimensional vector (x, y).

It is useful to introduce another representation of complex numbers, namely polar coordinates (r, θ):

$$x = r\cos\theta \qquad y = r\sin\theta \qquad (r \geq 0) \tag{1.1.3}$$

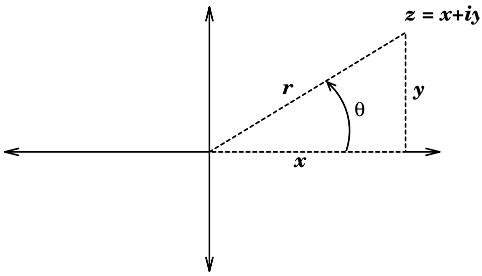

Fig. 1.1.1. The complex plane ("z plane")

Hence the complex number z can be written in the alternative polar form:

$$z = x + iy = r(\cos\theta + i\sin\theta) \tag{1.1.4}$$

The radius r is denoted by

$$r = \sqrt{x^2 + y^2} \equiv |z| \tag{1.1.5a}$$

(note: $\equiv$ denotes equivalence) and naturally gives us a notion of the **absolute value** of z, denoted by $|z|$, that is, it is the length of the vector associated with z. The value $|z|$ is often referred to as the **modulus** of z. The angle θ is called the argument of z and is denoted by $\arg z$. When $z \neq 0$, the values of θ can be found from Eq. (1.1.3) via standard trigonometry:

$$\tan\theta = y/x \tag{1.1.5b}$$

where the quadrant in which x, y lie is understood as given. We note that $\theta \equiv \arg z$ is **multivalued** because $\tan\theta$ is a periodic function of θ with period π. Given $z = x + iy$, $z \neq 0$ we identify θ to have one value in the interval $\theta_0 \leq \theta < \theta_0 + 2\pi$, where θ_0 is an arbitrary number; others differ by integer multiples of 2π. We shall take $\theta_0 = 0$. For example, if $z = -1 + i$, then $|z| = r = \sqrt{2}$ and $\theta = \frac{3\pi}{4} + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. The previous remarks apply equally well if we use the polar representation about a point $z_0 \neq 0$. This just means that we translate the origin from $z = 0$ to $z = z_0$.

At this point it is convenient to introduce a special exponential function. The polar exponential is defined by

$$\cos\theta + i\sin\theta = e^{i\theta} \tag{1.1.6}$$

Hence Eq. (1.1.4) implies that z can be written in the form

$$z = re^{i\theta} \tag{1.1.4$'$}$$

This exponential function has all of the standard properties we are familiar with in elementary calculus and is a special case of the complex exponential function to be introduced later in this chapter. For example, using well-known trigonometric identities, Eq. (1.1.6) implies

$$e^{2\pi i} = 1 \qquad e^{\pi i} = -1 \qquad e^{\frac{\pi i}{2}} = i \qquad e^{\frac{3\pi i}{2}} = -i$$

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)} \qquad (e^{i\theta})^m = e^{im\theta} \qquad (e^{i\theta})^{1/n} = e^{i\theta/n}$$

With these properties in hand, one can solve an equation of the form

$$z^n = a = |a|e^{i\phi} = |a|(\cos\phi + i\sin\phi), \qquad n = 1, 2, \ldots$$

Using the periodicity of $\cos\phi$ and $\sin\phi$, we have

$$z^n = a = |a|e^{i(\phi+2\pi m)} \qquad m = 0, 1, \ldots, n-1$$

and find the n roots

$$z = |a|^{1/n} e^{i(\phi+2\pi m)/n} \qquad m = 0, 1, \ldots, n-1.$$

For $m \geq n$ the roots repeat.

If $a = 1$, these are called the n roots of unity: $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = e^{2\pi i/n}$. So if $n = 2$, $a = -1$, we see that the solutions of $z^2 = -1 = e^{i\pi}$ are $z = \{e^{i\pi/2}, e^{3i\pi/2}\}$, or $z = \pm i$. In the context of real numbers there are no solutions to $z^2 = -1$, but in the context of complex numbers this equation has two solutions. Later in this book we shall show that an nth-order polynomial equation, $z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$, where the coefficients $\{a_j\}_{j=0}^{n-1}$ are complex numbers, has n and only n solutions (roots), counting multiplicities (for example, we say that $(z-1)^2 = 0$ has two solutions, and that $z = 1$ is a solution of multiplicity two).

The **complex conjugate** of z is defined as

$$\bar{z} = x - iy = re^{-i\theta} \tag{1.1.7}$$

Two complex numbers are said to be equal if and only if their real and imaginary parts are respectively equal; namely, calling $z_k = x_k + iy_k$, for $k = 1, 2$, then

$$z_1 = z_2 \quad \Rightarrow \quad x_1 + iy_1 = x_2 + iy_2 \quad \Rightarrow \quad x_1 = x_2,\ y_1 = y_2$$

Thus $z = 0$ implies $x = y = 0$.

Addition, subtraction, multiplication, and division of complex numbers follow from the rules governing real numbers. Thus, noting $i^2 = -1$, we have

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2) \tag{1.1.8a}$$

and

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \qquad (1.1.8b)$$

In fact, we note that from Eq. (1.1.5a)

$$z\overline{z} = \overline{z}z = (x + iy)(x - iy) = x^2 + y^2 = |z|^2 \qquad (1.1.8c)$$

This fact is useful for division of complex numbers,

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} \\ &= \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} \end{aligned} \qquad (1.1.8d)$$

It is easily shown that the commutative, associative, and distributive laws of addition and multiplication hold.

Geometrically speaking, addition of two complex numbers is equivalent to that of the parallelogram law of vectors (see Figure 1.1.2).

The useful analytical statement

$$||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2| \qquad (1.1.9)$$

has the geometrical meaning that no side of a triangle is greater in length than the sum of the other two sides – hence the term for inequality Eq. (1.1.9) is the **triangle inequality**.

Equation (1.1.9) can be proven as follows.

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + \overline{z_1}z_2 \\ &= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z_2}) \end{aligned}$$

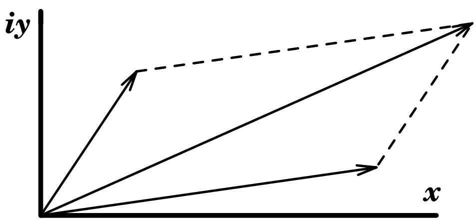

Fig. 1.1.2. Addition of vectors

Hence

$$|z_1 + z_2|^2 - (|z_1| + |z_2|)^2 = 2(\mathrm{Re}(z_1\overline{z_2}) - |z_1||z_2|) \leq 0 \qquad (1.1.10)$$

where the inequality follows from the fact that

$$x = \mathrm{Re}\, z \leq |z| = \sqrt{x^2 + y^2}$$

and $|z_1\overline{z_2}| = |z_1||z_2|$.

Equation (1.1.10) implies the right-hand inequality of Eq. (1.1.9) after taking a square root. The left-hand inequality follows by redefining terms. Let

$$W_1 = z_1 + z_2 \qquad W_2 = -z_2$$

Then the right-hand side of Eq. (1.1.9) (just proven) implies that

$$|W_1| \leq |W_1 + W_2| + |-W_2|$$

$$\text{or} \qquad |W_1| - |W_2| \leq |W_1 + W_2|$$

which then proves the left-hand side of Eq. (1.1.9) if we assume that $|W_1| \geq |W_2|$; otherwise, we can interchange W_1 and W_2 in the above discussion and obtain

$$||W_1| - |W_2|| = -(|W_1| - |W_2|) \leq |W_1 + W_2|$$

Similarly, note the immediate generalization of Eq. (1.1.9)

$$\left|\sum_{j=1}^{n} z_j\right| \leq \sum_{j=1}^{n} |z_j|$$

Problems for Section 1.1

1. Express each of the following complex numbers in polar exponential form:

$$\text{(a) } 1 \qquad \text{(b) } -i \qquad \text{(c) } 1+i$$

$$\text{(d) } \frac{1}{2} + \frac{\sqrt{3}}{2}i \qquad \text{(e) } \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

2. Express each of the following in the form $a + bi$, where a and b are real:

$$\text{(a) } e^{2+i\pi/2} \qquad \text{(b) } \frac{1}{1+i} \qquad \text{(c) } (1+i)^3 \qquad \text{(d) } |3+4i|$$

(e) Define $\cos(z) = (e^{iz} + e^{-iz})/(2)$, and $e^z = e^x e^{iy}$.
Evaluate $\cos(i\pi/4 + c)$, where c is real

3. Solve for the roots of the following equations:

$$\text{(a)}\ z^3 = 4 \qquad \text{(b)}\ z^4 = -1$$

$$\text{(c)}\ (az+b)^3 = c, \text{ where } a, b, c > 0 \qquad \text{(d)}\ z^4 + 2z^2 + 2 = 0$$

4. Estabilish the following results:

$$\text{(a)}\ \overline{z+w} = \bar{z} + \bar{w} \qquad \text{(b)}\ |z-w| \le |z| + |w| \qquad \text{(c)}\ z - \bar{z} = 2i\,\text{Im}\, z$$

$$\text{(d)}\ \text{Re}z \le |z| \qquad \text{(e)}\ |w\bar{z} + \bar{w}z| \le 2|wz| \qquad \text{(f)}\ |z_1 z_2| = |z_1||z_2|$$

5. There is a partial correspondence between complex numbers and vectors in the plane. Denote a complex number $z = a + bi$ and a vector $\mathbf{v} = a\hat{\mathbf{e}}_1 + b\hat{\mathbf{e}}_2$, where $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are unit vectors in the horizontal and vertical directions. Show that the laws of addition $z_1 \pm z_2$ and $\mathbf{v}_1 \pm \mathbf{v}_2$ yield equivalent results as do the magnitudes $|z|^2$, $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$. (Here $\mathbf{v} \cdot \mathbf{v}$ is the usual vector dot product.) Explain why there is no general correspondence for laws of multiplication or division.

1.2 Elementary Functions and Stereographic Projections

1.2.1 Elementary Functions

As a prelude to the notion of a function we present some standard definitions and concepts. A circle with center z_0 and radius r is denoted by $|z - z_0| = r$. A **neighborhood** of a point z_0 is the set of points z for which

$$|z - z_0| < \epsilon \tag{1.2.1}$$

where ϵ is some (small) positive number. Hence a neighborhood of the point z_0 is all the points inside the circle of radius ϵ, not including its boundary. An annulus $r_1 < |z - z_0| < r_2$ has center z_0, with inner radius r_1 and outer radius r_2. A point z_0 of a set of points $\mathcal{S}$ is called an **interior point** of $\mathcal{S}$ if there is a neighborhood of z_0 entirely contained within $\mathcal{S}$. The set $\mathcal{S}$ is said to be an **open set** if all the points of $\mathcal{S}$ are interior points. A point z_0 is said to be a **boundary point** of $\mathcal{S}$ if every neighborhood of $z = z_0$ contains at least one point in $\mathcal{S}$ and at least one point not in $\mathcal{S}$.

A set consisting of all points of an open set and none, some or all of its boundary points is referred to as a **region**. An open region is said to be **bounded** if there is a constant $M > 0$ such that all points z of the region satisfy $|z| \le M$, that is, they lie within this circle. A region is said to be **closed** if it contains all of its boundary points. A region that is both closed and bounded is called

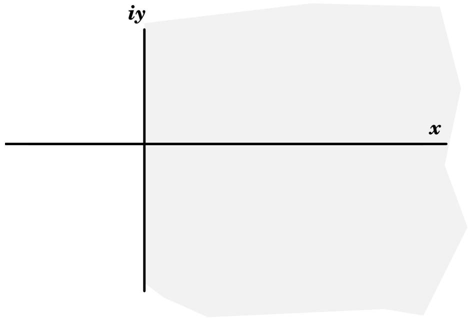

Fig. 1.2.1. Half plane

compact. Thus the region $|z| \leq 1$ is compact because it is both closed and bounded. The region $|z| < 1$ is open and bounded. The half plane $\operatorname{Re} z > 0$ (see Figure 1.2.1) is open and unbounded.

Let $z_1, z_2, \ldots, z_n$ be points in the plane. The $n - 1$ line segments $\overline{z_1 z_2}$, $\overline{z_2 z_3}$, $\ldots$, $\overline{z_{n-1} z_n}$ taken in sequence form a broken line. An open region is said to be **connected** if any two of its points can be joined by a broken line that is contained in the region. (There are more detailed definitions of connectedness, but this simple one will suffice for our purposes.) For an example of a connected region see Figure 1.2.2.)

A disconnected region is exemplified by all the points interior to $|z| = 1$ and exterior to $|z| = 2$: $\mathcal{S} = \{z : |z| < 1, |z| > 2\}$.

A connected open region is called a **domain**. For example the set (see Figure 1.2.3)

$$\mathcal{S} = \{z = re^{i\theta} : \theta_0 < \arg z < \theta_0 + \alpha\}$$

is a domain that is unbounded.

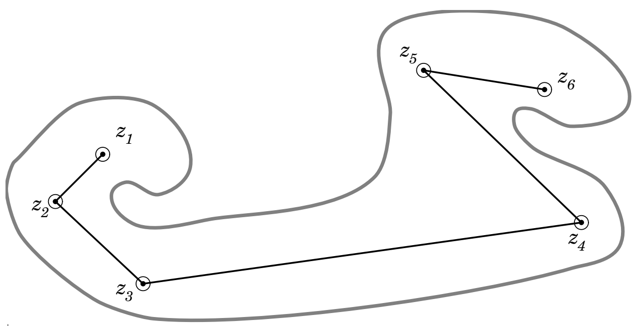

Fig. 1.2.2. Connected region

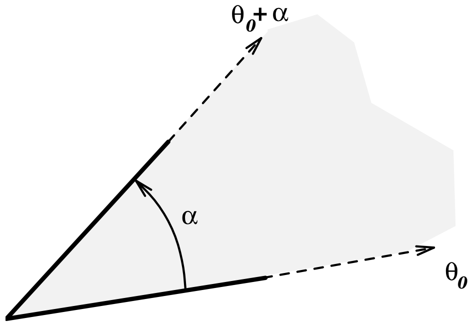

Fig. 1.2.3. Domain – a sector

Because a domain is an open set, we note that no boundary point of the domain can lie in the domain. Notationally, we shall refer to a region as $\mathcal{R}$; the closed region containing $\mathcal{R}$ and all of its boundary points is sometimes referred to as $\overline{\mathcal{R}}$. If $\mathcal{R}$ is closed, then $\mathcal{R} = \overline{\mathcal{R}}$. The notation $z \in \mathcal{R}$ means z is a point contained in $\mathcal{R}$. Usually we denote a domain by $\mathcal{D}$.

If for each $z \in \mathcal{R}$ there is a unique complex number $w(z)$ then we say $w(z)$ is a **function** of the complex variable z, frequently written as

$$w = f(z) \tag{1.2.2}$$

in order to denote the function f. Often we simply write $w = w(z)$, or just w. The totality of values $f(z)$ corresponding to $z \in \mathcal{R}$ constitutes the **range** of $f(z)$. In this context the set $\mathcal{R}$ is often referred to as the **domain of definition** of the function f. While the domain of definition of a function is frequently a domain, as defined earlier for a set of points, it does not need to be so.

By the above definition of a function we disallow multivaluedness; no more than one value of $f(z)$ may correspond to any point $z \in \mathcal{R}$. In Sections 2.2 and 2.3 we will deal explicitly with the notion of multivaluedness and its ramifications.

The simplest function is the **power** function:

$$f(z) = z^n, \qquad n = 0, 1, 2, \ldots \tag{1.2.3}$$

Each successive power is obtained by multiplication $z^{m+1} = z^m z$, $m = 0, 1, 2, \ldots$ A **polynomial** is defined as a linear combination of powers

$$P_n(z) = \sum_{j=0}^{n} a_j z^j = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \tag{1.2.4}$$

where the a_j are complex numbers (i.e.,[1] $a_j \in \mathbb{C}$). Note that the domain of definition of $P_n(z)$ is the entire z plane simply written as $z \in \mathbb{C}$. A **rational** function is a ratio of two polynomials $P_n(z)$ and $Q_m(z)$, where $Q_m(z) = \sum_{j=0}^{m} b_j z^j$

$$R(z) = \frac{P_n(z)}{Q_m(z)} \tag{1.2.5}$$

and the domain of definition of $R(z)$ is the z plane, *excluding* the points where $Q_m(z) = 0$. For example, the function $w = 1/(1+z^2)$ is defined in the z plane excluding $z = \pm i$. This is written as $z \in \mathbb{C} \setminus \{i, -i\}$.

In general, the function $f(z)$ is complex and when $z = x + iy$, $f(z)$ can be written in the complex form:

$$w = f(z) = u(x, y) + i\, v(x, y) \tag{1.2.6}$$

The function $f(z)$ is said to have the real part u, $u = \text{Re}\, f$, and the imaginary part v, $v = \text{I}m\, f$. For example,

$$w = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$

which implies

$$u(x, y) = x^2 - y^2 \qquad \text{and} \qquad v = 2xy.$$

As is the case with real variables we have the standard operations on functions. Given two functions $f(z)$ and $g(z)$, we define addition, $f(z) + g(z)$, multiplication $f(z)g(z)$, and composition $f[g(z)]$ of complex functions.

It is convenient to define some of the more common functions of a complex variable – which, as with polynomials and rational functions, will be familiar to the reader.

Motivated by real variables, $e^{a+b} = e^a e^b$, we define the exponential function

$$e^z = e^{x+iy} = e^x e^{iy}$$

Noting the polar exponential definition (used already in section 1.1, Eq. (1.1.6))

$$e^{iy} = \cos y + i \sin y$$

we see that

$$e^z = e^x(\cos y + i \sin y) \tag{1.2.7}$$

[1] Hereafter these abbreviations will frequently be used: i.e. = that is; e.g. = for example.