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978-0-521-53391-1 - An Introduction to Ordinary Differential Equations

James C. Robinson

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AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

This refreshing, introductory textbook covers standard techniques for solving ordinary differential equations, as well as introducing students to qualitative methods such as phase-plane analysis. The presentation is concise, informal yet rigorous; it can be used for either one-term or one-semester courses.

Topics such as Euler's method, difference equations, the dynamics of the logistic map and the Lorenz equations, demonstrate the vitality of the subject, and provide pointers to further study. The author also encourages a graphical approach to the equations and their solutions, and to that end the book is profusely illustrated. The MATLAB files used to produce many of the figures are provided in an accompanying website.

Numerous worked examples provide motivation for, and illustration of, key ideas and show how to make the transition from theory to practice. Exercises are also provided to test and extend understanding; full solutions for these are available for teachers.

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PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa
<http://www.cambridge.org>

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First published 2004

Printed in the United Kingdom at the University Press, Cambridge

Typeface Times 11/14 pt *System* L^AT_EX 2_ε [TB]

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication data

Robinson, James C. (James Cooper), 1969–
An introduction to ordinary differential equations / James C. Robinson.
p. cm.

Includes bibliographical references and index.

ISBN 0 521 82650 0 – ISBN 0 521 53391 0 (paperback)

1. Differential equations. I. Title

QA372.R77 2003
515'.352 – dc21 2003055186

ISBN 0 521 82650 0 hardback
ISBN 0 521 53391 0 paperback

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[More information](#)

To
Mum and Dad,
for all their love, help and support.

Contents

<i>Preface</i>	<i>page</i> xiii
Introduction	1
Part I First order differential equations	3
1 Radioactive decay and carbon dating	5
1.1 Radioactive decay	5
1.2 Radiocarbon dating	6
Exercises	8
2 Integration variables	9
3 Classification of differential equations	11
3.1 Ordinary and partial differential equations	11
3.2 The order of a differential equation	13
3.3 Linear and nonlinear	13
3.4 Different types of solution	14
Exercises	16
4 *Graphical representation of solutions using MATLAB	18
Exercises	21
5 ‘Trivial’ differential equations	22
5.1 The Fundamental Theorem of Calculus	22
5.2 General solutions and initial conditions	25
5.3 Velocity, acceleration and Newton’s second law of motion	29
5.4 An equation that we cannot solve explicitly	32
Exercises	33

Some of the chapters, and some sections within other chapters, are marked with an asterisk (*). These parts of the book contain material that either is more advanced, or expands on points raised elsewhere in the text.

6	Existence and uniqueness of solutions	38
6.1	The case for an abstract result	38
6.2	The existence and uniqueness theorem	40
6.3	Maximal interval of existence	41
6.4	The Clay Mathematics Institute's \$1 000 000 question	42
	Exercises	44
7	Scalar autonomous ODEs	46
7.1	The qualitative approach	46
7.2	Stability, instability and bifurcation	48
7.3	Analytic conditions for stability and instability	49
7.4	Structural stability and bifurcations	50
7.5	Some examples	50
7.6	The pitchfork bifurcation	54
7.7	Dynamical systems	56
	Exercises	56
8	Separable equations	59
8.1	The solution 'recipe'	59
8.2	The linear equation $\dot{x} = \lambda x$	61
8.3	Malthus' population model	62
8.4	Justifying the method	64
8.5	A more realistic population model	66
8.6	Further examples	68
	Exercises	72
9	First order linear equations and the integrating factor	75
9.1	Constant coefficients	75
9.2	Integrating factors	76
9.3	Examples	78
9.4	Newton's law of cooling	79
	Exercises	86
10	Two 'tricks' for nonlinear equations	89
10.1	Exact equations	89
10.2	Substitution methods	94
	Exercises	97
Part II	Second order linear equations with constant coefficients	99
11	Second order linear equations: general theory	101
11.1	Existence and uniqueness	101
11.2	Linearity	102
11.3	Linearly independent solutions	104
11.4	*The Wronskian	106

<i>Contents</i>		ix
11.5	*Linear algebra	107
	Exercises	109
12	Homogeneous second order linear equations	111
12.1	Two distinct real roots	112
12.2	A repeated real root	113
12.3	No real roots	115
	Exercises	118
13	Oscillations	120
13.1	The spring	120
13.2	The simple pendulum	122
13.3	Damped oscillations	123
	Exercises	126
14	Inhomogeneous second order linear equations	131
14.1	Complementary function and particular integral	131
14.2	When $f(t)$ is a polynomial	133
14.3	When $f(t)$ is an exponential	135
14.4	When $f(t)$ is a sine or cosine	137
14.5	Rule of thumb	139
14.6	More complicated functions $f(t)$	139
	Exercises	140
15	Resonance	141
15.1	Periodic forcing	141
15.2	Pseudo resonance in physical systems	145
	Exercises	148
16	Higher order linear equations	150
16.1	Complementary function and particular integral	150
16.2	*The general theory for n th order equations	152
	Exercises	153
Part III	Linear second order equations with variable coefficients	157
17	Reduction of order	159
	Exercises	162
18	*The variation of constants formula	164
	Exercises	168
19	*Cauchy–Euler equations	170
19.1	Two real roots	171
19.2	A repeated root	171
19.3	Complex roots	173
	Exercises	174

20	*Series solutions of second order linear equations	176
20.1	Power series	176
20.2	Ordinary points	178
20.3	Regular singular points	183
20.4	Bessel's equation	187
	Exercises	195
Part IV	Numerical methods and difference equations	199
21	Euler's method	201
21.1	Euler's method	201
21.2	An example	203
21.3	*MATLAB implementation of Euler's method	204
21.4	Convergence of Euler's method	206
	Exercises	209
22	Difference equations	213
22.1	First order difference equations	213
22.2	Second order difference equations	215
22.3	The homogeneous equation	215
22.4	Particular solutions	219
	Exercises	222
23	Nonlinear first order difference equations	224
23.1	Fixed points and stability	224
23.2	Cobweb diagrams	225
23.3	Periodic orbits	226
23.4	Euler's method for autonomous equations	227
	Exercises	230
24	The logistic map	233
24.1	Fixed points and their stability	234
24.2	Periodic orbits	234
24.3	The period-doubling cascade	237
24.4	The bifurcation diagram and more periodic orbits	238
24.5	Chaos	240
24.6	*Analysis of $x_{n+1} = 4x_n(1 - x_n)$	242
	Exercises	245
Part V	Coupled linear equations	247
25	*Vector first order equations and higher order equations	249
25.1	Existence and uniqueness for second order equations	251
	Exercises	252
26	Explicit solutions of coupled linear systems	253
	Exercises	257

<i>Contents</i>		xi
27	Eigenvalues and eigenvectors	259
27.1	Rewriting the equation in matrix form	259
27.2	Eigenvalues and eigenvectors	260
27.3	*Eigenvalues and eigenvectors with MATLAB	266
	Exercises	267
28	Distinct real eigenvalues	269
28.1	The explicit solution	270
28.2	Changing coordinates	271
28.3	Phase diagrams for uncoupled equations	276
28.4	Phase diagrams for coupled equations	279
28.5	Stable and unstable manifolds	281
	Exercises	282
29	Complex eigenvalues	285
29.1	The explicit solution	285
29.2	Changing coordinates and the phase portrait	287
29.3	The phase portrait for the original equation	291
	Exercises	292
30	A repeated real eigenvalue	295
30.1	\mathbb{A} is a multiple of the identity: stars	295
30.2	\mathbb{A} is not a multiple of the identity: improper nodes	295
	Exercises	299
31	Summary of phase portraits for linear equations	301
31.1	*Jordan canonical form	301
	Exercises	305
Part VI	Coupled nonlinear equations	307
32	Coupled nonlinear equations	309
32.1	Some comments on phase portraits	309
32.2	Competition of species	310
32.3	Direction fields	311
32.4	Analytical method for phase portraits	314
	Exercises	322
33	Ecological models	323
33.1	Competing species	323
33.2	Predator-prey models I	331
33.3	Predator-prey models II	334
	Exercises	338
34	Newtonian dynamics	341
34.1	One-dimensional conservative systems	341
34.2	*A bead on a wire	344

Cambridge University Press

978-0-521-53391-1 - An Introduction to Ordinary Differential Equations

James C. Robinson

Frontmatter

[More information](#)

xii

Contents

34.3	Dissipative systems	347
	Exercises	350
35	The ‘real’ pendulum	352
35.1	The undamped pendulum	352
35.2	The damped pendulum	356
35.3	Alternative phase space	358
	Exercises	358
36	*Periodic orbits	360
36.1	Dulac’s criterion	360
36.2	The Poincaré–Bendixson Theorem	361
	Exercises	362
37	*The Lorenz equations	364
38	What next?	373
38.1	Partial differential equations and boundary value problems	373
38.2	Dynamical systems and chaos	374
	Exercises	375
	<i>Appendix A Real and complex numbers</i>	379
	<i>Appendix B Matrices, eigenvalues, and eigenvectors</i>	382
	<i>Appendix C Derivatives and partial derivatives</i>	387
	<i>Index</i>	395

Preface

The aim of this book is to deal with all of the elementary methods for obtaining explicit solutions of ordinary differential equations, and then to introduce the ideas of qualitative analysis using phase plane techniques. Simple difference equations are also included, since their methods of solution are similar to those for linear differential equations. As well as being, I hope, an internally consistent choice of material, this selection of topics also has the advantage of preparing a student for a basic course on dynamical systems.

The book arose from my unsuccessful efforts to find a suitable text to recommend when I taught the first year Warwick differential equations course. Although there are a number of well-established and successful textbooks that treat this subject (these are discussed, along with other possibilities for further reading, in the final chapter), they seem either to include a large amount of additional material, or to concentrate only on the more advanced topics. I therefore produced a detailed set of lecture notes, which, with the encouragement of Alan Harvey and David Tranah at Cambridge University Press, eventually became this book. My thanks here to all those students who made helpful suggestions while this book was still at the lecture note stage.

Part I contains an informal discussion of the issues of existence and uniqueness of solutions, and treats the standard classes of first order differential equations that can be solved explicitly, as well as covering exact equations and substitution methods.

The first chapter of Part II shows that two linearly independent solutions are needed in order to solve the general homogeneous problem, and also contains a brief treatment of the Wronskian. The remainder of this section treats equations with constant coefficients, concentrating for the most part on the second order case, with higher order equations discussed briefly at the end.

Second order equations with non-constant coefficients are treated in Part III,

which covers reduction of order, the method of variation of constants, and series solutions.

Part IV turns aside from differential equations, motivating the study of difference equations by discussing Euler's method of numerical solution. Constant coefficient linear difference equations are covered, and then there are two chapters devoted to nonlinear difference equations. One of these goes beyond the confines of an introductory course and discusses the dynamics of the logistic map in some detail.

Part V treats coupled systems of two linear differential equations, starting with the substitution method that reduces the problem to a second order differential equation in one variable, the most reliable way to find explicit solutions. The remainder of this portion of the book deals with the matrix approach, showing how a calculation of the eigenvalues and eigenvectors of an appropriate matrix is enough to draw the phase portrait. This is done by changing to a coordinate system in which the equation is put into a standard form, providing an illustration of the Jordan canonical form of a matrix.

Part VI uses the methods from Part V in order to draw the phase plane diagrams for a variety of nonlinear systems, with examples taken from mathematical ecology and simple one-dimensional particle systems, including the pendulum. The book ends with a brief discussion of Dulac's criterion and the Poincaré–Bendixson Theorem, a chapter that investigates the complicated dynamics of the Lorenz Equations, and suggestions for further reading.

In addition to those already mentioned above I would like to thank various people who have contributed to this book. I first learned much of the material here from Tristram Jones-Parry at Westminster School, to whom much belated thanks for all his fine teaching many years ago. I also owe a debt of gratitude to all those who taught the course at Warwick before me, shaping its contents and therefore those of this book; in particular, I had useful guidance from the course notes of Alan Newell and Claude Baesens. I am most grateful to Andrew Stuart, who, in encouraging me to emphasise the links with linear algebra, made me fond of a subject that I still remembered with a shudder from my own undergraduate days. Thanks too to James Macdonald, whose 'Swarm of flies' program for his MMath project on the Lorenz equations was the inspiration behind Figure 37.8.

Over the past two months I have been able to think of little except phase planes and drawing figures in MATLAB: my wife, Tania Styles, has managed to endure my many variations on 'come and see this picture of a washing machine' with a smile. Heartfelt thanks to her for this, and, of course, for everything.

Finally, I would particularly like to thank my Ph.D. student, Oliver Tearne, and my father, John, both of whom read this book extremely carefully and made a number of very helpful comments. For whatever imperfections remain, my apologies to them and to my readers.