Introduction

Differential equations date back to the mid-seventeenth century, when calculus was discovered independently by Newton (*c.* 1665) and Leibniz (*c.* 1684). Modern mathematical physics essentially started with Newton's *Principia* (published in 1687) in which he not only developed the calculus but also presented his three fundamental laws of motion that have made the mathematical modelling of physical phenomena possible.[1]

Historically, advances in the theory of differential equations have come from the insights gained when trying to treat specific physical models. Despite this somewhat piecemeal development, the subject has become a well-defined and coherent area of mathematics. This book adopts a theoretical point of view, developing the theory to the point at which it can no longer be described as 'basic differential equations' and is about to become entangled with more advanced topics from the theory of dynamical systems. Of course, applications are used throughout to serve as motivation and illustration, but the emphasis is on a clean presentation of the mathematics.

You may find that some of the problems covered in the first few chapters are already familiar. The methods of solving these problems are well established, and you may be well practised at applying them. However, we will take care here to show why these methods work; giving proper justification of the methods can take some time, but as mathematicians we should not be satisfied merely with a set of 'recipes'. Nevertheless, knowing something about the details should not stop you from applying the methods you know already; rather you should be able to use them with more confidence.

Some of the chapters, and some sections within other chapters, are marked with an asterisk (*). These parts of the book contain either material that is more advanced, or material that expands on points raised elsewhere; while they could be omitted in the interests of brevity, they are intended to give some indication of the richness of the subject beyond the confines of an introductory course.

[1] Various modern editions of this work are available, translated from its original Latin.

There are three appendices, covering background material that is necessary at various points in the book. While some of this is elementary and may already be familiar (Appendix A recalls some notation and various facts about real and complex numbers that will be used throughout the book) some is a little more advanced. Problems with timetabling often mean that certain undergraduate courses have to rely on material that is yet to be taught in others, hence there are appendices on matrices, eigenvalues and eigenvectors (Appendix B) and on derivatives, partial derivatives and Taylor series (Appendix C). The calculation of eigenvalues and eigenvectors is treated in detail in the main part of the book.

The use of mathematical computer packages is now a standard part of the undergraduate curriculum, and an important tool in the armoury of practising mathematicians, scientists and engineers. Although the emphasis in the text is on pencil and paper analysis, and the book in no way relies on the availability of such software, some topics, particularly the treatment of coupled nonlinear equations using phase plane ideas in Chapters 28–37, can benefit greatly from the graphical possibilities modern computers provide. Almost all of the figures in this book have been generated using MATLAB, and very occasionally particular MATLAB commands are mentioned in the text. Nevertheless, it should be possible to carry out the numerical exercises suggested here using any of the major commercially available mathematical packages; and with a little more ingenuity using any programming language with graphical capabilities. The MATLAB files used to produce some of the figures, and mentioned in certain of the exercises, are available for download from the web at `www.cambridge.org/0521533910`.

There is no better way to learn this material than by working through a selection of examples. One set of examples is included in what is, I hope, a natural way in the text, with the end of each worked solution marked with a box (□). Another set of examples is given in the exercises that end each chapter, and these should be considered an integral part of the book. The majority consist of sample problems that can be treated with the methods of the chapter – in order to give teachers a reasonable choice of problems, there are intentionally more of these than you could reasonably be expected to do. Others, labelled with a 'T', are more theoretical and designed to give an indication of some of the mathematical issues raised, but not treated in detail, in the text. Finally, those exercises labelled with a 'C' are intended to encourage the use of the computer to perform routine calculations and investigate equations and their solutions graphically. Those involved in teaching courses based on this book may obtain copies of solutions to these exercises by applying to the publisher by email (`solutions@cambridge.org`).

I would welcome any comments or suggestions, either by post to the Mathematics Institute, University of Warwick, Coventry, CV4 7AL, U.K. or by email to `jcr@maths.warwick.ac.uk`; any errata that arise will be posted on my own website `www.maths.warwick.ac.uk/~jcr/IntroODEs.html`.

Part I

First order differential equations

1

Radioactive decay and carbon dating

Before we start our formal treatment of the subject we will look at a very simple example that nonetheless exhibits the power of differential equations as models of reality. One point to bear in mind in this chapter is the distinction to be made between finding the solution of a differential equation, and interpreting this solution.

1.1 Radioactive decay

Let $N(t)$ denote the number of radioactive atoms in some sample of material at time t. Then with $k > 0$ the equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -kN \tag{1.1}$$

is a very good model for the way that the number of radioactive atoms decays (see Exercise 1.1).

Although we will see later how to solve this equation, for now we will assume that when there are N_s isotopes at time s, the solution is

$$N(t) = N_s \mathrm{e}^{-k(t-s)}. \tag{1.2}$$

You can check that we really do have the solution: when $t = s$ the formula in (1.2) gives $N(s) = N_s$, while we have

$$\frac{\mathrm{d}}{\mathrm{d}t} N(t) = -kN_s \mathrm{e}^{-k(t-s)} = -kN(t),$$

and so the differential equation (1.1) is satisfied.

It follows from (1.2) that the number of radioactive isotopes decays exponentially to zero. Graphs of the solution for various values of $N(0)$, showing this decay, are plotted in Figure 1.1.

The *half-life* of a particular radioactive isotope is the time it takes for half of the radioactive isotopes to decay, and this is related to the constant k that appears in

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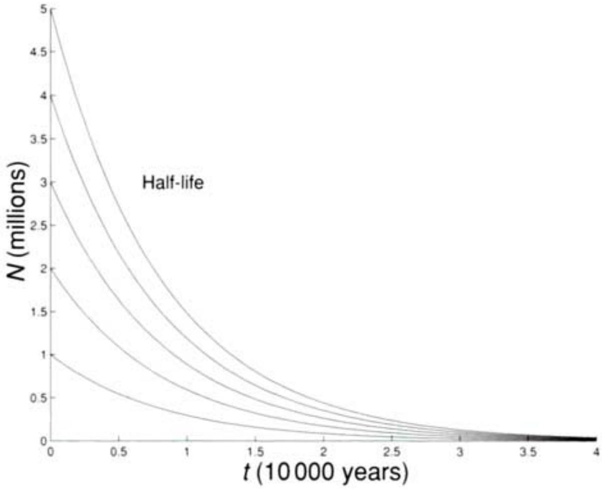

Fig. 1.1. Graph showing the number $N(t)$ of radioactive atoms falling off as a function of time, for a number of different values of N_0; the constant k is that for radioactive carbon 14. The half-life, approximately 5700 years, is marked by a dashed vertical line.

the equation. To find this relationship, suppose that there are N_0 radioactive atoms at time $t = 0$. Then the solution of (1.1) is

$$N(t) = N_0 e^{-kt}.$$

Half of the atoms will have decayed by time $t_{\rm half}$ when $N(t_{\rm half}) = \frac{1}{2}N_0$, i.e.

$$N_0 e^{-kt_{\rm half}} = \tfrac{1}{2}N_0 \qquad \Rightarrow \qquad e^{-kt_{\rm half}} = \tfrac{1}{2}.$$

Taking the (natural) logarithm of both sides gives

$$-kt_{\rm half} = -\ln 2,$$

and so the half-life is given by $t_{\rm half} = (\ln 2)/k$. Note that this time does not depend on the initial number of radioactive atoms.

1.2 Radiocarbon dating

The solution (1.2) forms the basis of the technology of radiocarbon dating. The essence of the method is as follows. Living matter is constantly taking up carbon from the air. The result is that within such material the ratio of the number of isotopes of radioactive carbon 14 (^{14}C) to the number of isotopes of stable carbon 12 (^{12}C) is essentially constant. Once the specimen is dead (for example, a tree is cut down for its wood, or cotton is harvested for weaving), the radioactive ^{14}C atoms begin to decay according to the model (1.1). Since the half-life of carbon 14 is approximately 5700 years, we need to take the constant k in (1.1) to be

$$k = \frac{\ln 2}{5700} \approx 1.216 \times 10^{-4}.$$

By examining the ratio of the number of isotopes of carbon 12 to carbon 14 in a sample of the material that we want to date, it is possible to work out the proportion remaining of the ^{14}C atoms that were initially present. Suppose that the sample stopped taking up carbon from the air when $t = s$, and that the number of ^{14}C atoms present then was N_s. If we know that the sample now (at time t_0) contains only a fraction p of the initial level of ^{14}C, then $N(t_0) = pN_s$.

Using our explicit solution $N(t) = N_s e^{-k(t-s)}$, we should have

$$pN_s = N(t_0) = N_s e^{-k(t_0-s)}.$$

Cancelling the factor of N_s in the two outside terms yields the equation

$$p = e^{-k(t_0-s)}.$$

Taking logarithms of both sides we have

$$\ln p = -k(t_0 - s),$$

and so the year s from which the sample dates is given by

$$s = t_0 + \frac{\ln p}{k}. \tag{1.3}$$

In 1988, the Shroud of Turin (see Figure 1.2) was dated by three independent groups of scientists from Arizona, Oxford and Zurich. Fibres from the shroud were

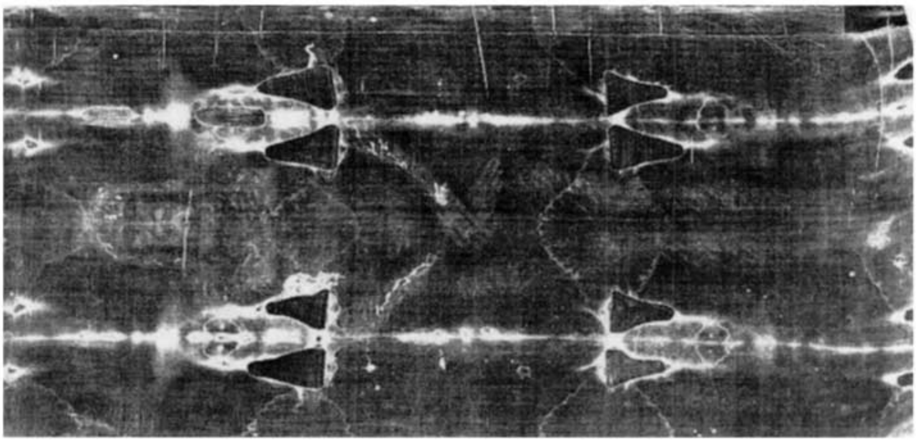

Fig. 1.2. The Shroud of Turin: carbon dated to the fourteenth century. Photograph © 1978 Barrie M. Schwortz (his website at `www.shroud.com` is well worth a visit).

found to contain about 92% of the level in living matter.[1] Using the expression in (1.3) shows that the Shroud therefore dates from

$$s = 2003 + \frac{\ln 0.92}{0.000\,121\,6} \approx 1318,$$

putting its origin squarely in the Middle Ages.

Exercises

1.1 Radioactive isotopes decay at random, with a fixed probability of decay per unit time. Over a time interval Δt, suppose that the probability of any one isotope decaying is $k\Delta t$. If there are N isotopes, how many will decay on average over a time interval Δt? Deduce that

$$N(t + \Delta t) - N(t) \approx -Nk\Delta t,$$

and hence that $\mathrm{d}N/\mathrm{d}t = -kN$ is an appropriate model for radioactive decay.

1.2 Plutonium 239, virtually non-existent in nature, is one of the radioactive materials used in the production of nuclear weapons, and is a by-product of the generation of power in a nuclear reactor. Its half-life is approximately 24 000 years. What is the value of k that should be used in (1.1) for this isotope?

1.3 In 1947 a large collection of papyrus scrolls, including the oldest known manuscript version of portions of the Old Testament, was found in a cave near the Dead Sea; they have come to be known as the 'Dead Sea Scrolls'. The scroll containing the book of Isaiah was dated in 1994 using the radiocarbon technique;[2] it was found to contain between 75% and 77% of the initial level of carbon 14. Between which dates was the scroll written?

1.4 A large round table hangs on the wall of the castle in Winchester. Many would like to believe that this is the Round Table of King Arthur, who (so legend would have it) was at the height of his powers in about AD 500. If the table dates from this time, what proportion of the original carbon 14 would remain? In 1976 the table was dated using the radiocarbon technique, and 91.6% of the original quantity of carbon 14 was found.[3] From when does the table date?

1.5 Radiocarbon dating is an extremely delicate process. Suppose that the percentage of carbon 14 remaining is known to lie in the range $0.99p$ to $1.01p$. What is the range of possible dates for the sample?

[1] P. E. Damon *et al.*, 'Radiocarbon dating of the Shroud of Turin', *Nature* **337** (1989), 611–615.
[2] A. J. Jull *et al.*, 'Radiocarbon dating of the scrolls and linen fragments from the Judean Desert', *Radiocarbon* **37** (1995), 11–19.
[3] M. Biddle, *King Arthur's Round Table* (Boydell Press, 2001).

2

Integration variables

Because of the intimate relationship between differentiation and integration (discussed in more detail in the next chapter) there will be many integrals in this book, and it is worth pausing now in order to make sure that we have an appropriately unambiguous notation for integrals.

Although in theory mathematicians make careful distinctions between 'the function f' and '$f(x)$', the value that f takes at a particular point x, this distinction is rarely maintained in day-to-day informal discussions.

Usually this does not cause any trouble. However, consider the following problem, posed in 'everyday' language:

Find the area under the graph of $f(x)$ between a and x.

Although the meaning of this is clear, 'find the shaded area in Figure 2.1', there is some potential for confusion when we try to write this down mathematically, since there are too many xs around. Converting the English into symbols gives

$$\int_a^x f(x)\,\mathrm{d}x, \tag{2.1}$$

and it should be clear that this is not satisfactory, since the symbol x is used in two different ways: once as the upper limit of the range of integration ($\int_a^x$), and once as the variable that is being integrated over ($\mathrm{d}x$).

When we integrate a function between two limits, for example[1]

$$\int_a^b f(x)\,\mathrm{d}x,$$

the variable that we are integrating over is a 'dummy' variable. It is just there to tell us how to do the integration, and plays no rôle in the final answer, which will

[1] Observe that there is no need to change our notation for this particular definite integral, since no confusion can arise as to the rôle of x.

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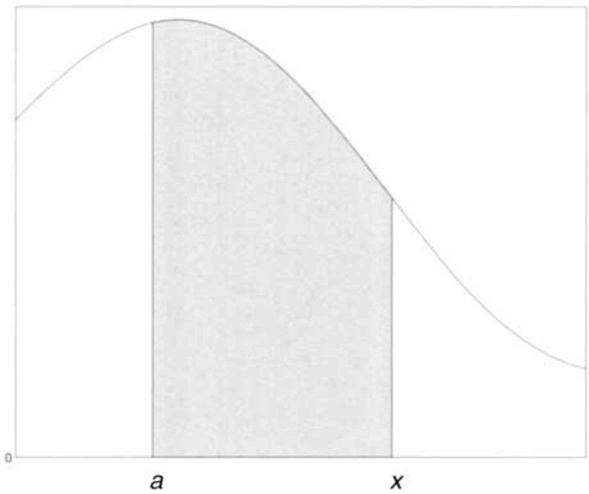

Fig. 2.1. 'Find the shaded area'.

only depend on a and b. So

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(\theta)\,\mathrm{d}\theta = \int_a^b f(\aleph)\,\mathrm{d}\aleph.$$

(We can change the name of the dummy variable with no effect on the integral.)

The obvious solution, then, is to change the integration variable in (2.1) to something other than x. However, changing the variable to something completely different from x is likely to be confusing. The approach we will adopt will be to add a tilde $\sim$ to the integration variable, so that instead of (2.1) we will write

$$\int_a^x f(\tilde{x})\,\mathrm{d}\tilde{x}. \tag{2.2}$$

All being well this should keep things 'clean' but should not be too jarring.

We will also do something similar when evaluating integrals where x is an upper limit, i.e.

$$\int_a^x f(\tilde{x})\,\mathrm{d}\tilde{x} = \Big[F(\tilde{x})\Big]_{\tilde{x}=a}^{x},$$

when $F' = f$.

Of course, very few people are this careful when they are doing calculations and the backs of mathematicians' envelopes are full of things like (2.1) rather than the pedantic (2.2).

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3

Classification of differential equations

Before we begin we need to introduce a simple classification of differential equations which will let us increase the complexity of the problems we consider in a systematic way.

3.1 Ordinary and partial differential equations

The most significant distinction is between ordinary and partial differential equations, and this depends on whether ordinary or partial derivatives occur.

Partial derivatives cannot occur when there is only one independent variable. The independent variables are usually the arguments of the function that we are trying to find, e.g. x in $f(x)$, t in $x(t)$, both x and y in $G(x, y)$. The most common independent variables we will use are x and t, and we will adopt a special shorthand for derivatives with respect to these variables: we will use a dot for $\mathrm{d}/\mathrm{d}t$, so that

$$\dot{z} = \frac{\mathrm{d}z}{\mathrm{d}t} \qquad \text{and} \qquad \ddot{z} = \frac{\mathrm{d}^2 z}{\mathrm{d}t^2};$$

and a prime symbol for $\mathrm{d}/\mathrm{d}x$, so that

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} \qquad \text{and} \qquad y'' = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$

Usually we will prefer to use time as the independent variable.

In an ordinary differential equation (ODE) there is only one independent variable, for example the variable x in the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x),$$