INTRODUCTORY TALK AT THE OPENING OF THE CONFERENCE

ANATOLE M. STEPIN

Dear colleagues and friends! This conference is devoted to the memory of Vladimir Mikhaylovich Alexeyev, professor of the Moscow State University, who untimely passed away in 1980.

Vladimir Mikhaylovich was one of the lecturers at Katsiveli Mathematical School in 1971. Such schools (and conferences) were regularly conducted by the Institute of Mathematics of the Academy of Sciences of Ukraine since 1963. This regularity was broken because of certain political changes in the former Republics of the USSR. I would like to express my hope that our meeting in Crimea is a step towards restoring Crimean mathematical schools and conferences.

The organizers of the present conference suggested that I give a talk today, on the first day of our work here, and tell you what I recall about Vladimir Mikhaylovich Alexeyev. I am grateful to the Organizational Committee for the invitation to participate in the conference and for the honor to present my recollections of V. M. Alexeyev, a brilliant mathematician and personality.

V. M. Alexeyev was born on June 17, 1932. His father comes from a well known family of Russian merchants, the Alexeyev family, who gave the world K. S. Stanislavsky, a famous reformer of theatrical art. While in the ninth grade, Volodya Alexeyev started attending Math Club meetings at the Moscow State University, and in the next year he was honored with the first prize at Moscow Mathematical Olympiad for high school students. He then entered the Moscow State University to major in Mechanics and Mathematics. A. N. Kolmogorov was his advisor. His Master's Diploma (1955) and Ph.D. Thesis (1959) were devoted to the rigorous proof of the possibility of satellite exchange in the three body problem – the phenomenon discovered for the first time with the help of numerical methods (L. Bekker, 1920).

My first meeting with Alexeyev took place in 1964 when I was a student. The point is that the students who were majoring at Mechanics and Mathematics were supposed to have pedagogical practice, semester-long one, and my advisor, Felix Alexandrovich Berezin, suggested that I practice at the Kolmogorov Boarding School in Physics and Mathematics, founded a year before. V. M. Alexeyev was one of the lecturers for ninth graders at this school. He was lecturing on Mathematical Analysis while we, his assistants, were conducting practical sessions with students on this and other subjects as well.

As a lecturer, V. M. Alexeyev was of course interested in his students thoroughly practicing in class all the theoretical material. He recommended problems appropriate for solving in class, yet at the same time never fixed the manner in which the practical sessions were actually conducted. Let me put it straight here – the teachers and students were using this freedom quite a lot. However sometimes, during the so-called "pedagogical meetings" and personal discussions, Vladimir Mikhaylovich kept us from excesses of the system. In general though, the main ingredient of the working atmosphere of the Kolmogorov School was the feeling of common creativity, "co-creativity", between teachers and students, and V. M. Alexeyev was among the most appropriate persons to help us to form such an atmosphere.

After finishing my practice as an assistant at the Kolmogorov School I received an offer from A. N. Kolmogorov who suggested that I continue teaching in the School. From that time on our pedagogical contacts with V. M. Alexeyev continued. Some of students of this school later became active participants in seminars on dynamical systems at the Moscow State University; let me name, e.g., A. Krygin, Yu. Osipov, Ya. Pesin, E. Sataev (by the way, Eugene Sataev participates in the present conference).

In 1964-1965 academic year V. M. Alexeyev and Ya. G. Sinai invited me to give a talk at their seminar in dynamical systems and ergodic theory (they were running this seminar after V. A. Rokhlin moved to Leningrad). The thing was that by that time, working at F. A. Berezin's seminar, I constructed a periodic Abelian group of mesure-preserving transformations whose maximal spectral type did not subordinate its convolution square.

The problem about group property of spectra of ergodic dynamical systems dates back to A. N. Kolmogorov; V. M. Alexeyev and Ya. G. Sinai at that time were actively interested in spectral theory of dynamical systems. The discussion at the seminar concerned my construction, in which both Alexeyev and Sinai participated, and certainly assisted in a much deeper understanding of the phenomenon which I discovered.

It turned out that the following fact was responsible for the spectral group property breaking (and, as it was found out later, in some other examples of unexpected or even unusual behavior of dynamical systems): the group of all automorphisms of the standard probability space, endowed with weak topology, is not complete with respect to one-sided uniformity (this circumstance had been noticed already by Halmos but was never used before). The points of the corresponding completion (or even compactification) are closely related to the notion of joining introduced later on. Especially important role in this relation is played by the joinings which are the limit points of Koopman operators corresponding to the dynamical systems.

In the fall of 1965 V. M. Alexeyev together with a large group of Moscow mathematicians participated in the famous Humsan conference. It took place in the village Humsan, close to the Tyan'-Shan' mountains and the city of

Tashkent. Three of us, Vladimir Mikhaylovich, A. Katok and myself, lived there in a big bright room of a comfortable mansion.

Our common accommodation in Humsan is still fresh in my memory, and I have a good reason for that. The point is that the experience in dealing with periodic transformation groups and the advice of F. A. Berezin not to stop research in this direction prompted me to study general dynamical systems as perturbations of a sequence of periodic transformations, and at that time I actively thought about the possibility of such an approach. A. Katok joined me in this at that time.

We used every opportunity to talk to V. M. Alexeyev about our (not yet embossed distinctly) ideas and preliminary arguments. He spared a lot of attention to us, in general approved the idea of approximation, made critical remarks and in some cases insistently requested formal proofs (though sometimes finding heuristical geometrical arguments satisfactory). Once V. M. made an interesting comparison of the newly born method of periodic approximations with the theory of approximation of functions. It so happened that Vladimir Mikhaylovich was one of the first who got acquainted with the initial outline of the method of periodic approximations; his friendly criticism was very helpful, and our paper, joint with A. Katok, published in Proceedings of the Academy of Sciences (1966) received a lot of attention from mathematicians.

Alexeyev himself at that time continued thinking over various questions about the asymptotic behaviour of motion in the three body problem. Here it would be appropriate to mention two stages in the research of final types of motions in this problem. The first stage is characterized by the usage of methods from perturbation theory developed precisely for these purposes. Alexeyev summarized the results of this stage in his paper published in the collection "The problems of movement of artificial celestial bodies" (1967).

In Humsan Vladimir Mikhaylovich was thinking about the existence of movements with "temporary capture", when a comet following a trajectory, co-asymptotic to a certain parabola, turns around the Sun prescribed number of times, despite passing another celestial body close by. He constructed examples of such movements with the help of the so-called discontinuous solutions of the ideal Kepler problem. Vladimir Mikhaylovich gave a talk about these results at the International Congress of Mathematicians in Moscow (1966). At the time he also started thinking about applications of the methods of symbolic dynamics and the theory of hyperbolic dynamical systems to the problem of classification of two-sided final movements in the three body problem.

Let me go back to the time of the Humsan conference. It should be said that the participants of this meeting managed to also have a good time during the conference. For example, Vladimir Mikhaylovich turned out to be a champion in swimming across the nearby mountain river with a very strong and fast current. Nadya Brushlinskaya, the wife of V.I.Arnol'd at the time, was very excited about these races praising Volodya Alexeyev for his success.

One more recollection about V. M. Alexeyev is also related to the same river in which he splashed quite often. Once, an acquaintance of mine, one of the participants, managed to dive in the river with his glasses on. And sure thing, the glasses fell off, and even though we both kept diving in the river and the water was crystal clear we could not find the glasses. Next morning I modeled the loss of the glasses: I made a model using aluminium wire, then dove and dropped "the glasses" exactly where they fell yesterday and observed their trajectory. Next to the spot where my model landed I found the true glasses in a perfect condition. When Vladimir Mikhaylovich learnt about this he told me rather seriously: "Tolya, you took into your heart the trouble of another man".

After the Humsan School Vladimir Mikhaylovich concentrated upon a complete solution of the two-sided version of the classification problem as regards asymptotic types of movements in the three body problem. The point is that the author of the classification of one-sided final types of movements, the French astronomer Chazy, formulated the statement (1929) on the coincidence of the final types as $t \to \pm\infty$. The existence of asymptotically symmetric movements of various types (covering all the possibilities) was established by Lagrange, Euler, Poincare, Birkhoff, Chazy, K. A. Sitnikov and V. I. Arnol'd.

The first rigorous result showing the possibility of asymptotic asymmetry of movements in three body problem was obtained by K. A. Sitnikov (1953). He implemented a partial capture (and, therefore, complete break-up), i.e. the possibility of combination of types: hyperbolic as $t \to -\infty$ and hyperbola-elliptic as $t \to +\infty$; with the help of numerical methods this was discovered earlier (1947) by O. Yu. Schmidt, well-known algebraist, polar explorer and the author of certain cosmogonical hypothesis. This achievement together with the aforementioned result of V. M. Alexeyev, concerning the possibility of satellite exchange, completely solved the problem of two-sided classification of final types in the case when the energy of the system is positive.

The main problem on the agenda then became the question about the possibility of complete capture (partial decay), i.e. the existence of movements of hyperbola-elliptical type as $t \to -\infty$ and bounded as $t \to +\infty$. This question called for the application to the three body problem the techniques of constructing and investigating (with the help of symbolic dynamics) of hyperbolic sets, developed in well-known works by D. V. Anosov, Ya. G. Sinai, S. Smale.

In 1968 Vladimir Mikhaylovich Alexeyev constructed an example of complete capture as well as examples of hyperbola-elliptical (or bounded) as $t \to -\infty$ and oscillating as $t \to +\infty$ (the latter means that for some pair of bodies the distance between them is unbounded but does not tend to infinity). Thus, it was established that all combinations of final types of movements can be realized (of course, taking into account the sign of the energy constant). This result as well as the classification of types of movements of one-dimensional oscillator in the force field, periodically depending on time, were presented in a series of papers published by Vladimir Mikhaylovich in Sbornik.Mathematics (1968-1969) and soon became famous.

Let me notice here that the existence of solutions to differential equations of second order having, in a sense, a random distribution of zeros, was observed earlier by Cartright, Littlewood and Levinson (1957). They constructed the solutions that admit coding by arbitrary sequences of zeros and ones in such a way that for some T_0, $T_1 \in \mathbb{R}^+$ zero (one) is associated with the interval between consecutive zeros of the corresponding solution, approximately equal in length to T_0 (respectively T_1).

Vladimir Mikhaylovich was invited to give a talk on his results at the International Congress of Mathematicians in Nice (1970). Shortly prior to the time of the Congress Jean Leray phoned to Alexeyev. Leray praised very highly the mathematical achievements of V. M. Alexeyev and asked him to be merciful to Chazy. It must be said that by the time the text of Alexeyev's talk have already been prepared and after mentioning Chazy's contribution the following was written: "C'est pour rendre hommage à cet éminent mathématicien et astronome français, dont les travaux ont stimulé en grande partie ce qui est expose ci-dessous, et aussi pour soulinger la continuité de l'effort des diverses generations de mathematiciens, que j'ai donné à cette conférence le titre même de deux de ses Mémoires". It so happend that V. M. Alexeyev was not included in the group of Soviet participants to the Congress, and he passed the text of the talk to me before my departure for Nice. At the Congress I reported about Alexeyev's results concerning final motions.

Let me point out a fact showing how focused V. M. Alexeyev was when he worked on the three body problem. It can be seen from his papers that he was a master of rigorous style, and at the same time, was capable to capture the attention of the reader. This made the work of the editor of translations of mathematical literature into Russian very important to him, and he devoted a lot of his time to this work. However, there was a clear seven-year (1963-1970) gap in this activity important especially for students; it was during this period the idea to attack the three body problem first ripened in V. M. Alexeyev's mind, and then was realized with the help of the new methods of dynamical systems theory.

V. M. Alexeyev actively participated in the life of Moscow Mathematical Society; he supervised the "Communications to MMS" section of the journal "Russian Mathematical Surveys" and gave talks at the meetings of MMS. The topic of one of his talks was the discussion of the recent result by Schweitzer who solved the Seifert problem on the nonexistence of closed trajectories for smooth vector fields on the 3-sphere. For several years V. M. Alexeyev was elected the Secretary of MMS. Once, in 1971 as far as I remember, Vladimir Mikhaylovich said to me: "You should also work in this capacity". I answered directly to him that for me at that time the main problem was to get visiting position abroad in order to earn money for buying an apartment for my family. Vladimir Mikhaylovich was sympathetic and took no offence at my de-facto refusal to his offer.

After coming back from visiting abroad I was in touch with V. M. Alexeyev regarding problems upon which our students were working (Yu. Osipov, S. Pidkujko, A. Tagi-Zade). The last time I visited Vladimir Mikhaylovich was in September of 1980 when I told him about the Mathematical School on Differential Equations which had been organized by the Institute of Mathematics of Ukrainian Academy of Sciences and Uzhgorod University and had taken place that summer in Carpathian mountains.

On December the 1st, 1980, Vladimir Mikhaylovich passed away. By the proposal of Ya. G. Sinai the annual lecture in memory of Alexeyev was established at the Mathematics Department of the Moscow State University. One of these lectures was delivered a few years ago by S. Smirnov, a colleague of Vladimir Mikhailovich and one of the members of the Organizing Committee of Moscow Mathematical Olympiads for high school students. I would like to end my talk quoting from that lecture by S. Smirnov.

"Alexeyev died when he was only 48, the age when Tolkien's hobbits reach their maturity. We do not know whether Alexeyev had found the time to read the Great Book of the Ring – it was not yet translated into Russian, but he read English easily and liked science fiction. If VMA read the biography of Frodo Baggins, then surely he must have felt his spiritual kinship to all the gentle-hobbits and to their creator..."

Department of Theory of Functions and Functional Analysis, Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, Vorob'evy Gory, Main Building MSU, Moscow 119899, Russia

E-mail address: stepin@mech.math.msu.su

Cambridge University Press
978-0-521-53365-2 — Topics in Dynamics and Ergodic Theory
Edited by Sergey Bezuglyi, Sergiy Kolyada
Excerpt
More Information

MINIMAL IDEMPOTENTS AND ERGODIC RAMSEY THEORY

VITALY BERGELSON

Contents

1. Introduction

What is common between the invertibility of distal maps, partition regularity of diophantine equation $x - y = z^2$, and the notion of mild mixing? The answer is: idempotent ultrafilters, and the goal of this survey is to convince the reader of the unifying role and usefulness of idempotent ultrafilters (and, especially, the minimal ones) in ergodic theory, topological dynamics and Ramsey theory.

We start with reviewing some basic facts about ultrafilters. The reader will find the missing details and more information in the self-contained Section 3 of [B2]. (See also [HiS] for a comprehensive presentation of the material related to topological algebra in the Stone-Čech compactification).

An ultrafilter p on $\mathbb{N} = \{1, 2, ...\}$ is, by definition, a *maximal filter*, namely, a nonempty family of subsets of $\mathbb{N}$ satisfying the following conditions (the first three of which constitute the definition of a *filter*):

(i) $\emptyset \notin p$;
(ii) $A \in p$ and $A \subset B$ imply $B \in p$;
(iii) $A \in p$ and $B \in p$ imply $A \cap B \in p$;
(iv) (maximality) if $r \in \mathbb{N}$ and $\mathbb{N} = A_1 \cup A_2 \cup ... \cup A_r$, then for some i, $1 \leq i \leq r$, $A_i \in p$.

The space of ultrafilters, denoted by $\beta\mathbb{N}$, and equipped with appropriately defined topology, is nothing but Stone-Čech compactification of $\mathbb{N}$ and plays

This work was partially supported by NSF under the grants DMS-9706057 and DMS-0070566.

an important role in various areas of mathematics including topology, analysis and ergodic Ramsey theory.

In what follows we will find it useful to view ultrafilters as finitely-additive, {0,1}-valued probability measures on the power set $\mathcal{P}(\mathbb{N})$.

Given an ultrafilter $p \in \beta\mathbb{N}$, define a mapping $\mu_p : \mathcal{P}(\mathbb{N}) \to \{0,1\}$ by $\mu_p(A) = 1 \Leftrightarrow A \in p$. It is easy to see that $\mu_p(\emptyset) = 0$, $\mu_p(\mathbb{N}) = 1$ (follows from (i), (iv) and (ii)), and that for any finite collection of disjoint sets $A_1, A_2, ..., A_r$, one has $\mu_p(\bigcup_{i=1}^r A_i) = \sum_{i=1}^r \mu_p(A_i)$. Indeed, note that if none of A_i belongs to p, then both sides equal zero. Also, it follows from (i) that at most one among the (disjoint!) sets A_i may satisfy $A \in p$, in which case both sides of the above equation equal one.

One of the major advantages of viewing the ultrafilters as measures is that one can naturally define the convolution operation which makes $\beta\mathbb{N}$ a compact semigroup. Given two σ-additive measures μ and ν on a topological group G, the convolution is usually defined as $\mu * \nu(A) = \int_G \mu(Ay^{-1})d\nu(y)$. In particular, $\mu * \nu(A) > 0$ iff for ν-many y one has $\mu(Ay^{-1}) > 0$. Taking into account that a value of ultrafilter measure on a set $A \subseteq \mathbb{N}$ is positive iff it equals one, we make the following definition in which for a reason to be explained in the remark below, we denote the convolution by $+$.

Definition 1.1. *Given $p, q \in \beta\mathbb{N}$, the convolution $p + q$ is defined by*

$$p + q = \{A \subseteq \mathbb{N} : \{n : (A - n) \in p\} \in q\}.$$

In other words, A is $(p+q)$-large iff the set $A - n = \{n \in \mathbb{N} : m + n \in A\}$ is p-large for q-many n.

It is not too hard to check that $p+q$ is an ultrafilter and that the operation defined above is associative (see, for example, [B2], p.27).

Now we shall explain the reason for denoting this operation by $+$. For any $n \in \mathbb{N}$ define an ultrafilter μ_n as a "delta measure" concentrated at point n:

$$\mu_n(A) = \begin{cases} 1, & n \in A \\ 0, & n \notin A. \end{cases}$$

The ultrafilters $\mu_n, n \in \mathbb{N}$, are called *principal* and it is clear that for any $n, k \in \mathbb{N}$ the convolution of μ_n and μ_k equals μ_{n+k}. In other words, the principal ultrafilters μ_n, $n \in \mathbb{N}$, form a semigroup which is isomorphic to $(\mathbb{N}, +)$ and the convolution defined above extends the operation $+$ to the space $\beta\mathbb{N}$, the closure of $\mathbb{N}$. At this point it will be instructive to say a few words about the topology on $\beta\mathbb{N}$. Given $A \subset \mathbb{N}$, let $\overline{A} = \{p \in \beta\mathbb{N} : A \in p\}$. It is immediate that for any $A, B \subset \mathbb{N}$ one has $\overline{A} \cap \overline{B} = \overline{A \cap B}$, $\overline{A} \cup \overline{B} = \overline{A \cup B}$. Also, since $\overline{\mathbb{N}} = \beta\mathbb{N}$, one has $\bigcup_{A \in \mathcal{A}} \overline{A} = \beta\mathbb{N}$, where $\mathcal{A} = \{\overline{A} : A \subset \mathbb{N}\}$. It follows that the set $\mathcal{A}$ forms the basis for the open sets of $\beta\mathbb{N}$ (and the basis for closed sets too!). One can show that with this topology $\beta\mathbb{N}$ is a compact Hausdorff space and that for any fixed $p \in \beta\mathbb{N}$ the function $\lambda_p(q) = p + q$ is a continuous self map of $\beta\mathbb{N}$ (see for example Theorems 3.1 and 3.2 in [B2]). In view of these facts, $(\beta\mathbb{N}, +)$ becomes a compact *left topological* semigroup. We remark in passing that the operation $\rho_p(q) = q+p$ is, unlike $\lambda_p(q)$, continuous only when p is a principal ultrafilter, and that the convolution defined above on $\beta\mathbb{N}$ is the unique extension of the operation $+$ on $\mathbb{N}$ such that $\lambda_p(q)$ and $\rho_p(q)$ have the properties described above.

Before going on to explore additional features of the semigroup $(\beta\mathbb{N}, +)$ that are important for us we want to caution the reader that while having various nice and convenient properties, the semigroup $(\beta\mathbb{N}, +)$ is in many respects an odd and counterintuitive object. First, the compact Hausdorff space $\beta\mathbb{N}$ is too large to be metrizable: its cardinality is that of $\mathcal{P}(\mathcal{P}(\mathbb{N}))$. Yet, in view of the fact that $\overline{\mathbb{N}} = \beta\mathbb{N}$, it is a closure of a countable set $\mathbb{N}$. Second, the operation $+$ on $\beta\mathbb{N}$ is highly non-commutative: the center of the semigroup $(\beta\mathbb{N}, +)$ contains only the principal ultrafilters. (Here the analogy with the convolution of σ-additive measures on locally compact abelian groups fails. The reason: the ultrafilters, being only *finitely* additive measures, do not obey the Fubini theorem which is crucial for the commutativity of the convolution of σ-additive measures).

By a theorem due to Ellis [E1], any compact semigroup with a left-continuous operation has an idempotent. Actually, $(\beta\mathbb{N}, +)$ has plenty of them, since any compact subsemigroup in $(\beta\mathbb{N}, +)$ should have one and there are 2^c disjoint compact subsemigroups in $\beta\mathbb{N}$. As we shall see below, of special importance for combinatorial and ergodic-theoretical applications are *minimal* idempotents, which we will define and apply later in this section. In a way, idempotent ultrafilters in $\beta\mathbb{N}$ are, in a way, just generalized shift-invariant measures. Indeed, if $p+p=p$, it means that any $A \in p = p+p$ has the property that $\{n : (A-n) \in p\} \in p$, or, in other words, for p-almost all n, the set $A-n$ is p-large.

It is easy to see that principal ultrafilters are never idempotent and hence, if p is an idempotent ultrafilter, any p-large set A is infinite, as is the p-large set $\{n : (A-n) \in p\}$. As we shall presently see, the members of idempotent ultrafilters always contain highly structured subsets which can be viewed as generalized subsemigroups of $\mathbb{N}$.

Let $A \in p$, where $p+p=p$. Since

$$A \cap \{n : (A-n) \in p\} \in p,$$

we can choose $n_1 \in A$ such that $A_1 = A \cap (A - n_1) \in p$. (Note that this is nothing but a version of Poincaré recurrence theorem; the important bonus is that $n_1 \in A$. By iterating this procedure one can chose $n_2 \in A \cap (A - n_1)$, $n_2 > n_1$, such that

$$A_1 \cap (A_1 - n_2) = A \cap (A - n_1) \cap (A - n_2) \cap (A - n_1 - n_2) \in p.$$

Note that $n_1, n_2, n_1 + n_2 \in A$. Continuing in this fashion, one obtains an increasing sequence $(n_i)_{i=1}^{\infty}$ and inductively defined sets $A = A_0, A_1, A_2, \ldots$, such that $n_1 \in A$, $n_{i+2} \in A_{i+1} := A_i \cap (A_i - n_{i+1})$, $i = 0, 1, 2, \ldots$. One readily checks that this construction implies that A contains the set of *finite sums* of $(n_i)_{i=1}^{\infty}$:

$$FS(n_i)_{i=1}^{\infty} = \{n_{i_1} + n_{i_2} + ... + n_{i_k},\ k \in \mathbb{N},\ i_1 < i_2 < ... < i_k\}.$$

Such sets of finite sums are customarily called IP sets (IP stands for IdemPotent) and are featured in the following important theorem due to N.Hindman [Hi1].

Theorem 1.2. (N.Hindman). *For any finite partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$ one of the cells of partition contains an IP set.*

Proof. Fix any idempotent ultrafilter $p \in \beta\mathbb{N}$ and observe that one (and only one!) of C_i belongs to it. Now use the fact proved above that any member of p contains an IP set. □

Let $\mathcal{F}$ denote the family of non-empty finite subsets of N. Noticing that the mapping $\mathcal{F} \to \mathbb{N}$ defined by $\{i_1, i_2, ..., i_k\} \to 2^{i_1} + 2^{i_2} + ... + 2^{i_k}$ is 1-1 and that elements of IP sets are naturally indexed by elements of $\mathcal{F}$, we have that each of the following two theorems implies Hindman's theorem, each revealing yet another facet of it.

Theorem 1.3. (Finite unions theorem). *For any finite partition $\mathcal{F} = \bigcup_{i=1}^{r} C_i$, one of C_i contains an infinite sequence of non-empty disjoint sets $(U_i)_{i\in\mathbb{N}}$ together with all the unions $U_{i_1} \cup U_{i_2} \cup ... \cup U_{i_k},\ i_1 < i_2 < ... < i_k, k \in \mathbb{N}$. In addition, one can assume without the loss of generality that for all $i \in \mathbb{N}$ one has $\max U_i < \min U_{i+1}$.*

Theorem 1.4. *For any finite partition of an IP set in $\mathbb{N}$ one of the cells of the partition contains an IP set.*

Exercise 1. Prove that Theorems 1.2, 1.3, 1.4 are equivalent.

In the proof of Hindman's theorem above IP sets emerge as subsets of members of idempotent ultrafilters. One may wonder whether given an idempotent p and a set $A \in p$, it is possible to find in A an IP set which is itself p-large. It turns out that this is not always the case. For example, the minimal idempotents which we will define below, can not have this property. The following theorem shows that, nevertheless, any IP set is a support of an idempotent.

Theorem 1.5. *For any sequence $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ there is an idempotent $p \in \beta\mathbb{N}$ such that $FS((x_i)_{i\in\mathbb{N}}) \in p$.*

Sketch of the proof. Let $\Gamma = \bigcap_{n=1}^{\infty} \overline{FS((x_i)_{i=n}^{\infty})}$. (The closures are taken in the natural topology of $\beta\mathbb{N}$). Clearly, Γ is compact and non-empty. It is not hard to show that Γ is a subsemigroup of $(\beta\mathbb{N}, +)$. Being a compact left-topological semigroup, Γ has an idempotent. If $p \in \Gamma$ is an idempotent, then $\overline{\Gamma} = \Gamma \ni p$