Introduction

> It must be admitted that the use of geometric intuition has no logical necessity in mathematics, and is often left out of the formal presentation of results. If one had to construct a mathematical brain, one would probably use resources more efficiently than creating a visual system. But the system is there already, it is used to great advantage by human mathematicians, and it gives a special flavor to human mathematics.
>
> *Ruelle (1999)*

Higher-dimensional category theory is the study of a zoo of exotic structures: operads, n-categories, multicategories, monoidal categories, braided monoidal categories, and more. It is intertwined with the study of structures such as homotopy algebras (A_∞-categories, L_∞-algebras, Γ-spaces, ...), n-stacks, and n-vector spaces, and draws it inspiration from areas as diverse as topology, quantum algebra, mathematical physics, logic, and theoretical computer science.

No surprise, then, that the subject has developed chaotically. The rush towards formalizing certain commonly-imagined concepts has resulted in an extraordinary mass of ideas, employing diverse techniques from most of the subject areas mentioned. What is needed is a transparent, natural, and practical language in which to express these ideas.

The main aim of this book is to present one. It is the language of generalized operads. It is introduced carefully, then used to give simple descriptions of a variety of higher categorical structures.

I hope that by the end, the reader will be convinced that generalized operads provide as appropriate a language for higher-dimensional category theory as vector spaces do for linear algebra, or sheaves for algebraic geometry. Indeed, the reader may also come to share the feeling that generalized operads are as

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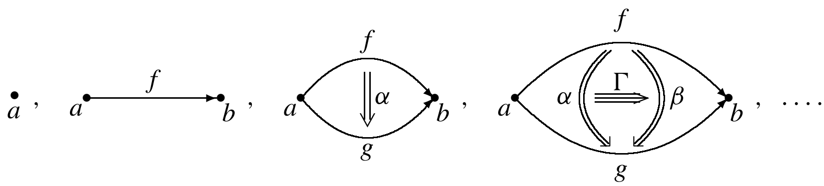
applicable and pervasive in mathematics at large as are n-categories, the usual focus of higher-dimensional category theorists.

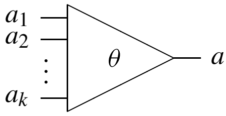
Here are some of the structures that we will study, presented informally.

Let $n \in \mathbb{N}$. An n-**category** consists of 0-**cells** (objects) $a, b, \ldots$, 1-**cells** (arrows) $f, g, \ldots$, 2-**cells** (arrows between arrows) $\alpha, \beta, \ldots$, 3-**cells** (arrows between arrows between arrows) $\Gamma, \Delta, \ldots$, and so on, all the way up to n-**cells**, together with various composition operations. The cells are usually drawn like this:



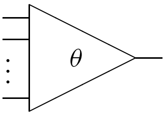
Typical example: for any topological space X there is an n-category whose k-cells are maps from the closed k-dimensional ball into X. A 0-category is just a set, and a 1-category just an ordinary category.

A **multicategory** consists of objects $a, b, \ldots$, arrows $\theta, \phi, \ldots$, a composition operation, and identities, just like an ordinary category, the difference being that the domain of an arrow is not just a single object but a finite sequence of them. An arrow is therefore drawn as



(where $k \in \mathbb{N}$), and composition turns a tree of arrows into a single arrow. Vector spaces and linear maps form a category; vector spaces and multilinear maps form a multicategory.

An **operad** is a multicategory with only one object. Explicitly, an operad consists of a set $P(k)$ for each $k \in \mathbb{N}$, whose elements are thought of as 'k-ary operations' and drawn as



with k input wires on the left, together with a rule for composing the operations and an identity operation. Example: for any vector space V there is an operad whose k-ary operations are the linear maps $V^{\otimes k} \longrightarrow V$.

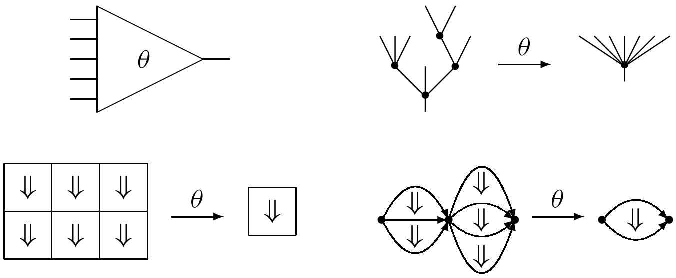


Fig. 0-A. Operations θ in four different types of generalized operad

Operads describe operations that take a finite sequence of things as input and produce a single thing as output. A finite sequence is a 1-dimensional entity, so operads can be used, for example, to describe the operation of composing a (1-dimensional) string of arrows in a (1-)category. But if we are interested in higher-dimensional structures such as n-categories then we need a more general notion of operad, one where the inputs of an operation can form a higher-dimensional shape – a grid, perhaps, or a tree, or a so-called pasting diagram. For each choice of 'input type' T, there is a class of **T-operads**. A T-operad consists of a family of operations whose inputs are of the specified type, together with a rule for composition; for instance, if the input type T is 'finite sequences' then a T-operad is an ordinary operad. Fig. 0-A shows typical operations θ in a T-operad, for four different choices of T. Similarly, there are **T-multicategories**, where the shapes at the domain and codomain of arrows are labelled with the names of objects. These are the 'generalized operads' and 'generalized multicategories' at the heart of this book.

The uniting feature of all these structures is that they are purely algebraic in definition, yet near-impossible to understand without drawing or visualizing pictures. They are inherently geometrical.

A notorious problem in this subject is the multiplicity of definitions of n-category. Something like a dozen different definitions have been proposed, and there are still very few precise results stating equivalence between any of them. This is not quite the scandal it may seem: it is hard to say what 'equivalence' should even mean. Suppose that Professors X and Y each propose a definition of n-category. To compare their definitions, you find a way of taking one of X's n-categories and deriving from it one of Y's n-categories, and *vice versa*, then you try to show that doing one process then the other gets you back to where you started. It is, however, highly unrealistic to expect that you will get back to *exactly* where you started. For most types of mathematical structure, getting back to somewhere isomorphic to your starting point would be a reasonable expectation. But for n-categories, as we shall see, this is still unrealistic: the canonical notion of equivalence of n-categories is much weaker than isomorphism. Finding a precise definition of equivalence for a given definition of n-category can be difficult. Indeed, many of the proposed definitions of n-category did not come with accompanying proposed definitions of equivalence, and this gap must be almost certainly be filled before any comparison results can be proved.

Is this all 'just language'? There would be no shame if it were: language can have the most profound effect. New language can make new concepts thinkable, and make old, apparently obscure, concepts suddenly seem natural and obvious. But there is no clear line between mathematical language and 'real' mathematics. For example, we will see that a 3-category with only one 0-cell and one 1-cell is precisely a braided monoidal category, and that the free braided monoidal category on one object is the sequence $(B_n)_{n\in\mathbb{N}}$ of braid groups. So if n-categories are just language, not 'real' mathematical objects, then the same is true of the braid groups, which describe configurations of knotted string. The distinction begins to look meaningless.

Here is a summary of the contents.

Motivation for topologists

Topology and higher-dimensional category theory are intimately related. The diagrams that one cannot help drawing when thinking about higher categorical structures can very often be taken literally as pieces of topology. We start with an informal discussion of the connections between the two subjects. This includes various topological examples of n-categories, and an account of how the world of n-categories is a mirror of the world of homotopy groups of spheres.

Part I. Background

We will build on various 'classical' notions. Those traditionally considered the domain of category theorists are in Chapter 1: ordinary categories, bicategories, strict n-categories, and enrichment. Classical operads and multicategories have Chapter 2 to themselves. They should be viewed as categorical structures too, although, anomalously, operads are best known to homotopy theorists and multicategories to categorical logicians.

The familiar concept of monoidal (tensor) category can be formulated in a remarkable number of different ways. We look at several in Chapter 3, and prove them equivalent. Monoidal categories can be identified with one-object 2-categories, so this is a microcosm of the comparison of different definitions of n-category.

Part II. Operads

This introduces the central idea of the text: that of generalized ('higher') operad and multicategory. The definitions – of generalized operad and multicategory, and of algebra for a generalized operad or multicategory – are stated and explained in Chapter 4, and some further theory is developed in Chapter 6.

There is a truly surprising theory of enrichment for generalized multicategories – it is not at all the routine extension of traditional enriched category theory that one might expect. This was to have formed Part IV of the book, but for reasons of space it was (reluctantly) dropped. A summary of the theory, with pointers to the original papers, is in Section 6.8.

The rest of Part II is made up of examples and applications. Chapter 5 is devoted to so-called **fc**-multicategories, which are generalized multicategories for a certain choice of input shape. They turn out to provide a clean setting for some familiar categorical constructions that have previously been encumbered by technical restrictions. In Chapter 7 we look at opetopic sets, structures analogous to simplicial sets and used in the definitions of n-category proposed by Baez, Dolan, and others. Again, the language of higher operads provides a very clean approach; we also find ourselves drawn inexorably into higher-dimensional topology.

Part III. n-categories

Using the language of generalized operads, some of the proposed definitions of n-category are very simple to state. We start by concentrating on one in particular, in which an n-category is defined as an algebra for a certain globular operad. A globular operad is a T-operad for a certain choice of input type T; the associated diagrams are complexes of discs, as in the last arrow θ of Fig. 0-A. Chapter 8 explains what globular operads are in pictorial terms. In Chapter 9 we choose a particular globular operad, define an n-category as an algebra for it, and explore the implications in some depth.

The many proposed definitions of n-category are not as dissimilar as they might at first appear. We go through most of them in Chapter 10, drawing together the common threads.

Appendices

This book is mostly about description: we develop language in which structures can be described simply and naturally, accurately reflecting their geometric reality. In other words, we mostly avoid the convolutions and combinatorial complexity often associated with higher-dimensional category theory. Where things run less smoothly, and in other situations where a lengthy digression threatens to disrupt the flow of the main text, the offending material is confined to an appendix. As long as a few plausible results are taken on trust, the entire main text can be read and understood without looking at any of the appendices.

A few words on terminology are needed. There is a distinction between 'weak' and 'strict' n-categories, as will soon be explained. For many years only the strict ones were considered, and they were known simply as 'n-categories'. More recently it came to be appreciated that weak n-categories are much more abundant in nature, and many authors now use 'n-category' to mean the weak version. I would happily join in, but for the following obstacle: in most parts of this book that concern n-categories, both the weak and the strict versions are involved and discussed in close proximity. It therefore seemed preferable to be absolutely clear and say either 'weak n-category' or 'strict n-category' on every occasion. The only exceptions are in this Introduction and the Motivation for topologists, where the modern convention is used.

The word 'operad' will be used in various senses. The most primitive kind of operad is an operad of sets without symmetric group action, and this is our starting point (Chapter 2). We hardly ever consider operads equipped with symmetric group actions, and when we do we call them 'symmetric operads'; see p. 58 for a more comprehensive warning.

Any finite sequence $x_1, \ldots, x_n$ of elements of a monoid has a product $x_1 \cdots x_n$. When $n = 0$, this is the unit element. Similarly, an identity arrow in a category can be regarded as the composite of a zero-length string of arrows placed end to end. I have taken the view throughout that there is nothing special about units or identities; they are merely nullary products or composites. Related to this is a small but important convention: the natural numbers, $\mathbb{N}$, start at zero.

Motivation for topologists

I'm a goddess *and* a nerd!
Bright (1999)

Higher-dimensional category theory *can* be treated as a purely algebraic subject, but that would be missing the point. It is inherently topological in nature: the diagrams that one naturally draws to illustrate higher-dimensional structures can be taken quite literally as pieces of topology. Examples of this are the braidings in a braided monoidal category and the pentagon appearing in the definitions of both monoidal category and A_∞-space.

This section is an informal description of what higher-dimensional category theory is and might be, and how it is relevant to topology. Grothendieck, for instance, suggested that tame topology should be the study of n-groupoids; others have hoped that an n-category of cobordisms between cobordisms between … will provide a clean setting for topological quantum field theory; and there is convincing evidence that the whole world of n-categories is a mirror of the world of homotopy groups of spheres.

There are no real theorems, proofs, or definitions here. But to whet your appetite, here is a question to which we will reach an answer by the end:

Question What is the close connection between the following two facts?

A No-one ever got into trouble for leaving out the brackets in a tensor product of several objects (abelian groups, etc.). For instance, it is safe to write $A \otimes B \otimes C$ instead of $(A \otimes B) \otimes C$ or $A \otimes (B \otimes C)$.

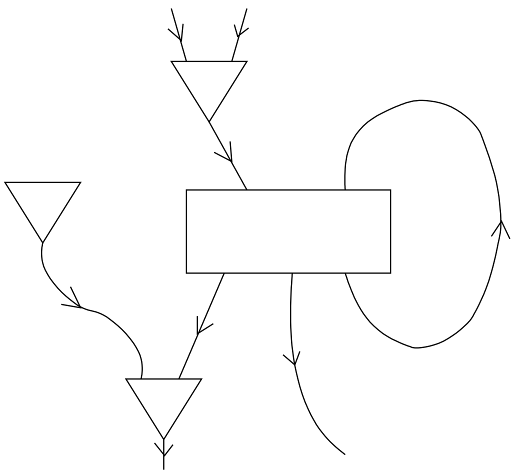
B There exist non-trivial knots in $\mathbb{R}^3$.

The very rough idea

In ordinary category theory we have diagrams of objects and arrows such as

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \;.$$

We can imagine more complex category-like structures in which there are diagrams such as



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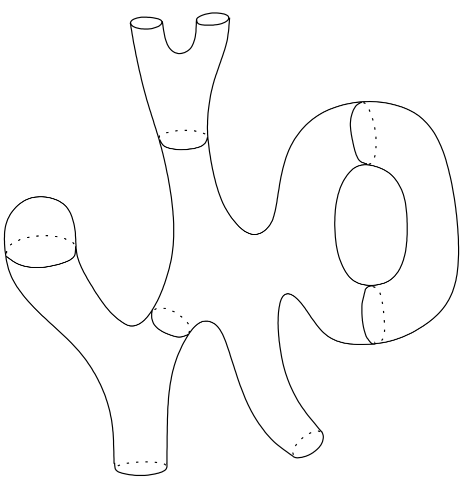
This looks like an electronic circuit diagram or a flow chart; the unifying idea is that of ‘information flow’. It can be redrawn as



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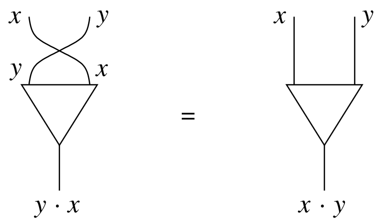
which looks like a surface or a diagram from topological quantum field theory.

We can also use diagrams like this to express algebraic laws such as commutativity:



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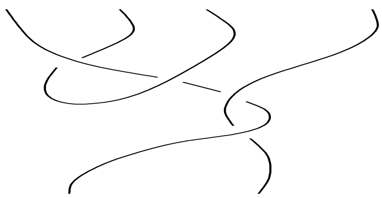
The fact that two-dimensional TQFTs are the same as commutative Frobenius algebras is an example of an explicit link between the spatial and algebraic aspects of diagrams like these.

Moreover, if we allow crossings, as in the commutativity diagram or as in



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then we obtain pictures looking like knots; and as we shall see, there are indeed relations between knot theory and higher categorical structures.

So the idea is:

Ordinary category theory uses 1-dimensional arrows $\longrightarrow$ Higher-dimensional category theory uses higher-dimensional arrows

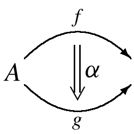
The natural topology of these higher-dimensional arrows is what makes higher-dimensional category theory an inherently topological subject.

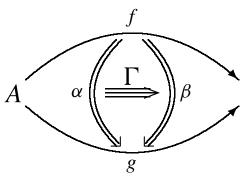
We will be concerned with structures such as operads, generalized operads (of which the variety familiar to homotopy theorists is a basic special case), multicategories, various flavours of monoidal categories, and n-categories; in this introduction I have chosen to concentrate on n-categories. Terminology: an n-category (or 'higher-dimensional category') is not a special kind of category, but a generalization of the notion of category; compare the usage of 'quantum group'. A 1-category is the same thing as an ordinary category, and a 0-category is just a set.

n-categories

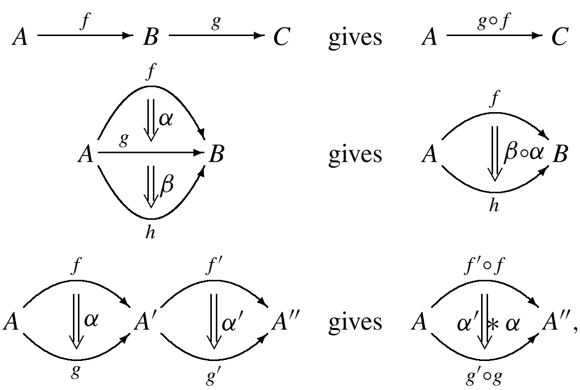
Here is a very informal

'Definition' Let $n \geq 0$. An n**-category** consists of

- **0-cells** or **objects**, A, B, ...
- **1-cells** or **morphisms**, drawn as $A \xrightarrow{f} B$
- **2-cells** 

('morphisms between morphisms')
- **3-cells** 

(where the arrow labelled Γ is meant to be going in a direction perpendicular to the plane of the paper)
- ...
- all the way up to n**-cells**
- various kinds of **composition**, e.g.



and so on in higher dimensions; and similarly **identities**.

These compositions are required to 'all fit together nicely' – a phrase hiding many subtleties. ω**-categories** (also known as ∞**-categories**) are defined similarly, by going on up the dimensions forever instead of stopping at n.

There is nothing forcing us to make the cells spherical here. We could, for instance, consider cubical structures, in which 2-cells look like