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Interfacial Fluid Dynamics

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1 Introduction

The Navier–Stokes equations for fluids that fill rigid containers govern an astonishing array of phenomena, in part due to the convective nonlinearity, $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$, in the velocity vector $\boldsymbol{v}$. The presence of this nonlinearity gives rise to non-uniqueness of the solutions, instability and bifurcation of solutions, the onset of turbulence, and separation. When the fluid is bounded by free boundaries, then the free-boundary nature of the problem introduces new nonlinearities that augment or compete with $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$. The shapes, positions, and evolution of the boundaries couple with the velocity fields and pressure p, and all of these must be determined simultaneously. Problems in interfacial fluid dynamics are intrinsically free-boundary problems.

The solutions of free-boundary problems are intrinsically non-unique. For example, consider the partially filled bucket of water shown in figure 1. In the absence of gravity and in a steady state the interface would take the shape of the arc of a sphere in which the interface intersects the wall at a contact line. If the contact line is fixed, this state is locally stable, in the sense that if the bucket were slightly shaken and then stopped, the velocity would decay to zero, and the interface would return to the shape shown. However, if the liquid is strongly disturbed, enough that a droplet is expelled, then another locally stable state would be created. It would consist of the bulk water plus a drop suspended in the passive gas. Clearly, there is infinite non-uniqueness with the possibility of an arbitrary number of drops. All of these states are allowable if the total water volume is preserved. Thus, to make an interfacial-fluids problem well posed one must fix the volume and specify the number of 'pieces' of material one wishes to examine. Generally, interfacial states are not globally stable since a concentrated disturbance of large enough amplitude can lead to the creation of a disconnected 'piece'.

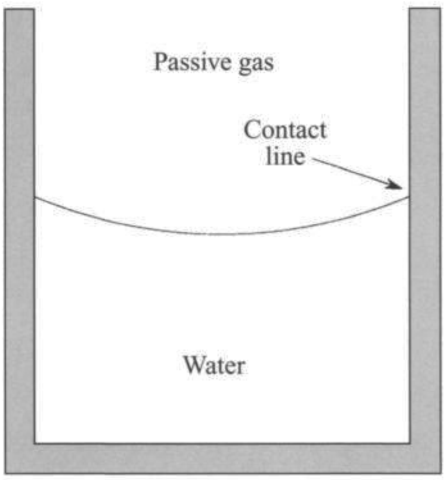

Figure 1. A sketch of a partially filled container in which the liquid–gas interface intersects the sidewall at a contact line.

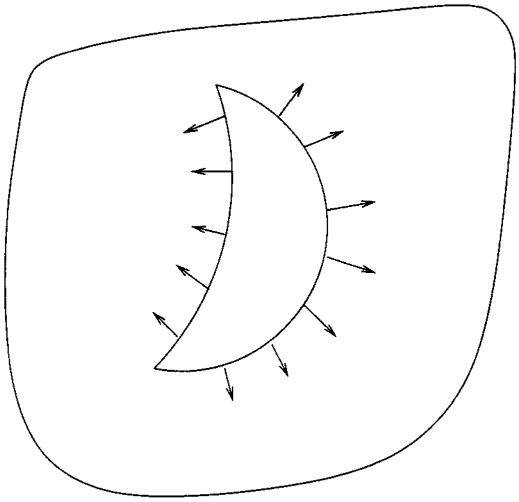

Figure 2. A sketch of an interface that is cut and opens due to the presence of surface tension.

The interface is generally taken to be a mathematical surface of zero thickness separating two phases. It is a free boundary but, furthermore, it is the site of localized forces, and is hence an *active boundary*. If one cuts an interface, as shown in figure 2, there is a force per unit length σ acting on the edge; σ is the surface or interfacial tension. The interface can, under certain circumstances, exhibit characteristics of surface viscosity and/or elasticity

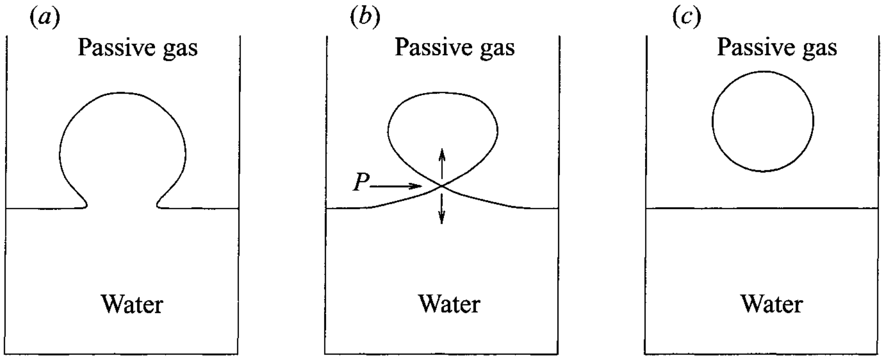

Figure 3. A sketch of the interface of figure 1 that is (*a*) given a strong disturbance at time $t = t_0$. (*b*) At $t = t_1$ the droplet contacts the bulk liquid at a single point P. (*c*) At $t = t_2$ long after the rupture, the disturbances have decayed, and the interfaces are now planar and spherical (or cylindrical in two dimensions) and locally stable.

independent of the properties of the bulk fluids. These interfacial forces can create interfacial motions, and, since the bulk liquid is viscous, these motions can be communicated to the bulk. Thus, bulk flow can drive the interface, and interfacial flow can drive the bulk; the two are tightly coupled.

The process of pinch-off of a droplet from the bulk is a subtle one that involves the rupture of a filament of liquid and the existence of a singularity at the point of splitting. Figure 3 shows the kinematics of such a scenario. As time t approaches a time $t_{1,}$ a single point, P, forms the bridge between the two bodies. The point P has two identities: bulk liquid and droplet liquid. At $t = t_1^+$, point P splits and is the site of *trajectory splitting* as shown in figure 3(*b*). Whenever trajectory splitting occurs, the flow field is locally singular.

Finally, if the liquid in the bucket of figure 1 is disturbed, the interface, and usually the contact lines as well, will move. Whenever there is mutual displacement (here gas and liquid) at a solid boundary, the displaced fluid must be removed and the displacing fluid must replace it. This process involves a more subtle version of trajectory splitting, and the contact-line singularity is non-integrable (since there are infinite forces required to move the line) unless the local conditions of no slip are remodelled.

There are countless applications of interfacial fluid mechanics. The coating of a substrate by a liquid, and the rupture and removal of liquid films from substrates are ubiquitous sub-processes in engineering and nature. Heat transfer devices often depend on a liquid film separating a vapour and a hot solid to protect the substrate from a toxic vapour or to control the rate of transfer of heat. This may involve evaporative effects and flows driven by surface-tension gradients. If the substrate is cold, efficient condensation would depend on the liquid being in dropwise form, running off the site, to free the surface for further condensation. The human lung is lined with a mucous layer that expels foreign particles as it flows, and that is a barrier to pollutants or aerosol drugs entering the blood stream (for example see Grotberg 1994). To spray paint a surface, one would wish to have individual droplets spread and merge into a continuous film, while if the droplets were contaminants, one would wish the droplets to 'ball up' so that a clean-up process could remove them.

This chapter first discusses how the model of a thin interface is rationalized. Then thin films and spreading drops are discussed in terms of dynamics and instability. Finally, the singularities inherent in rupture and coalescence are discussed.

2 Interfacial regions

Consider a system that is in static equilibrium and consists of two bulk phases that are in contact. A neighbourhood of this contact region, called the *interfacial region*, in which the fluids mix to some extent, has anomalous physical properties (e.g. density, pressure, etc.) compared to those of the bulk phases. These anomalies take the form of rapid variations in property values normal to the interfacial region. In physical systems of common (fluid) phases at room temperature, the thickness of the interfacial region Δ_s can range from as little as a fraction of a molecular diameter (a few Å) to, perhaps, ten molecular diameters. However, when the temperature is raised towards the critical temperature (at which a liquid and its vapour lose their identities and become indistinguishable), Δ_s becomes large and finally, in principle, infinite. Let us denote as L the length scale that characterizes the gross system, e.g. the smallest geometrical dimension of the container in which the system lies. The limit $\Delta_s/L \to 0$ gives great simplification of the mathematical models of interfacial fluid mechanics. The discussion that follows was strongly influenced by the developments in Quinn & Scriven (1970).

The *dividing-surface* approach, developed by Gibbs in 1876 (see his collected works, Gibbs 1948), involves the replacement of the actual system by two bulk fluids separated by a 'dividing surface' called the interface. The bulk properties are assumed to continue their definitions smoothly right up to the interface where they may experience jump discontinuities from one bulk to the other. The interface must then be endowed with surface properties appropriate to the physical system and constitutive assumptions on the surface material must be made. This technique simplifies the system, and also applies to situations where Δ_s is so small that the interfacial region cannot be expected to display a three-dimensional continuum character.

2.1 Interfacial conditions in the continuum model

Consider systems having two bulk phases and an interfacial region. As discussed earlier, the bulk phases are regarded as continuum fluids that have well-defined bulk properties Q. Likewise, the interfacial region is regarded as an interface of zero thickness that has well-defined surface properties Q_S in the interfaces. Call the two fluids I and II, and assume that as the interface $\mathscr{S}$ is approached from either side, the respective bulk properties have well-defined limits. Denote by $[Q]$ the jump in the bulk quantities Q, i.e.

$$[Q] = Q_{II} - Q_I \qquad \text{on} \qquad \mathscr{S}. \tag{2.1}$$

Likewise, require the unit normal vector $\boldsymbol{\nu}$ to $\mathscr{S}$ to be directed from II towards I. This convention on the normal vector is important to keep in mind since the mean curvature H of the interface, in particular its sign, is determined by $\boldsymbol{\nu}$.

Suppose, for example, that the Navier–Stokes equations hold in fluids I and II and that at $\mathscr{S}$ jump conditions are given. One could derive the interfacial jumps in their most general forms (Aris 1962; Slattery 1990). This would involve the derivation of transport theorems and the investigation of differential geometry. Rather than delve into these, only the conditions at the interface necessary for the explanation of the phenomena discussed will be derived here. In what follows in this section it is assumed that phase transformation is absent.

The kinematic condition gives the continuity of normal velocity across $\mathscr{S}$. If the bulk velocity field is $\boldsymbol{v} = (u, v, w)$ and V is the speed of the interface normal to itself, then $\boldsymbol{v} \cdot \boldsymbol{\nu} = V$, which for a single-valued front $\mathscr{S}: z = h(x, y, t)$, takes the form

$$w = h_t + uh_x + vh_y \qquad \text{on} \qquad z = h(x, y, t). \tag{2.2}$$

It is accepted on empirical grounds that, apart from cases of the flow of rarefied gases and certain instances of moving three-phase lines on solids (Dussan V. & Davis 1974), adjacent strata of viscous fluids have continuous tangential velocities. Therefore, on a fluid–fluid interface $\mathscr{S}$, the no-slip condition holds so that

$$[\boldsymbol{v} \cdot \boldsymbol{t}] = 0 \qquad \text{on} \qquad \mathscr{S}, \tag{2.3}$$

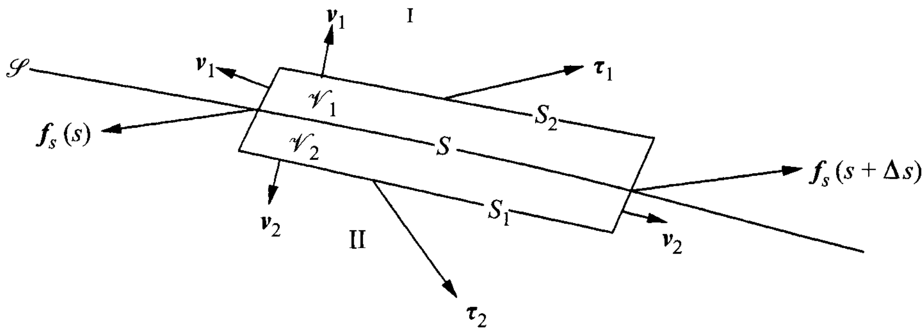

Figure 4. A sketch of a control volume $\mathscr{V}$ spanning an interface $\mathscr{S}$. The interfacial force f_s depends on arclength s, and $\boldsymbol{\nu}_i$ denote unit normals.

where $\boldsymbol{t}$ is the unit tangent vector to $\mathscr{S}$. The condition (2.3) holds on fluid–solid interfaces as well. However, when one wishes to use models involving inviscid fluids, condition (2.3) is abandoned.

Consider the balance of linear momentum in which there is a force per unit length $\boldsymbol{f}_s$ acting on points of the interface. Figure 4 shows in a two-dimensional system a segment of the interface and a control volume spanning $\mathscr{S}$. (In general the interface would be endowed with a mass per unit area γ_s; in what follows for the sake of simplicity, this is taken to be zero.) Here $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are stress vectors in the bulk fluids acting on areas S_1 and S_2 of the control volume, respectively. The interface $\mathscr{S}$ extends from s to $s+\Delta s$ where s is arclength, and the volume has width W_0 normal to the page. Finally, at the intersections of S_i with $\mathscr{S}$ there are localized interfacial forces $\boldsymbol{f}_s$.

Let us apply Newton's law to the material within $\mathscr{V} = \mathscr{V}_1 + \mathscr{V}_2$ and write for $i = 1, 2$, $\boldsymbol{\tau}_i = \boldsymbol{T}\cdot\boldsymbol{\nu}_i$, and unit normals $\boldsymbol{\nu}_i$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{V}} \rho \boldsymbol{v}\,\mathrm{d}\mathscr{V} = \int_{S_1+S_2} \boldsymbol{T}\cdot\boldsymbol{\nu}\,\mathrm{d}S + [\boldsymbol{f}_s(s+\Delta s) - \boldsymbol{f}_s(s)]\,W + \int_{\mathscr{V}} \rho \boldsymbol{F}\,\mathrm{d}\mathscr{V},$$

where $\boldsymbol{F}$ is the bulk body force per unit mass acting on fluids I and II, and $\boldsymbol{T}$ is the stress tensor. Since γ_s has been taken to be zero, there is no excess acceleration, or body force.

Now let the volume $\mathscr{V}$ collapse onto S and thus $\mathscr{V} \to 0$, $S_1, S_2 \to S$ and a patch of surface remains, as shown in figure 5. In this case $\boldsymbol{\nu}_1 \to \boldsymbol{n}$ and $\boldsymbol{\nu}_2 \to -\boldsymbol{n}$ on the sides I and II, respectively. If $\rho, \boldsymbol{v}, \boldsymbol{t}$ and $\boldsymbol{F}$ are smooth, then there is a local balance on $\mathscr{S}$,

$$-[\boldsymbol{T}\cdot\boldsymbol{n}]\,W_0\Delta s + [\boldsymbol{f}_s(s+\Delta s) + \boldsymbol{f}_s(s)]\,W_0 = 0.$$

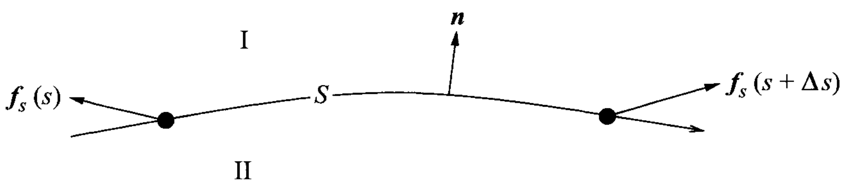

Figure 5. The control volume $\mathscr{V}$ of figure 4 becomes the surface S after the thickness approaches zero.

Now, divide by $W_0 \Delta s$ and let $\Delta s \to 0$ to yield the point balance on $\mathscr{S}$,

$$-[\boldsymbol{T} \cdot \boldsymbol{n}] + \frac{\partial \boldsymbol{f}_s}{\partial s} = 0. \tag{2.4}$$

In order to proceed further one needs a constitutive equation for the interfacial force $\boldsymbol{f}_s$.

If the interface is passive, then $\boldsymbol{f}_s = \boldsymbol{0}$ and the bulk stresses are continuous across $\mathscr{S}$. If there is surface tension σ only on $\mathscr{S}$, then $\boldsymbol{f}_s = \sigma \boldsymbol{t}$ so that $\boldsymbol{f}_s$ is tangent to $\mathscr{S}$ with magnitude σ; $-\sigma$ acts like an interfacial pressure. As a result,

$$\frac{\partial \boldsymbol{f}_s}{\partial s} = \sigma \frac{\partial \boldsymbol{t}}{\partial s} + \frac{\partial \sigma}{\partial s} \boldsymbol{t},$$

which has a tangential component only if the surface tension varies from position to position on $\mathscr{S}$. There is a normal component (Aris 1962) given by $\partial \boldsymbol{t}/\partial s = \kappa \boldsymbol{n}$ (a Frenet formula) and in the three-dimensional case $\partial \boldsymbol{t}/\partial s = 2H\boldsymbol{n}$, where H is the mean curvature of the surface. Hence, the balance of linear momentum (2.4) becomes

$$-[\boldsymbol{T} \cdot \boldsymbol{n}] + 2H\sigma \boldsymbol{n} + \frac{\partial \sigma}{\partial s} \boldsymbol{t} = \boldsymbol{0}. \tag{2.5a}$$

The normal component of (2.5*a*) is

$$[\boldsymbol{n} \cdot \boldsymbol{T} \cdot \boldsymbol{n}] = 2H\sigma, \tag{2.5b}$$

which for inviscid fluids is the well-known Laplace relation,

$$p_2 - p_1 = -2H\sigma. \tag{2.5c}$$

where $2H = -\nabla \cdot \boldsymbol{n}$. There is a pressure excess on the concave side of the interface of magnitude equal to surface tension times twice the mean curvature.

The tangential component of (2.5*a*) is

$$[\boldsymbol{t} \cdot \boldsymbol{T} \cdot \boldsymbol{n}] - \frac{\partial \sigma}{\partial s} = 0, \tag{2.5d}$$

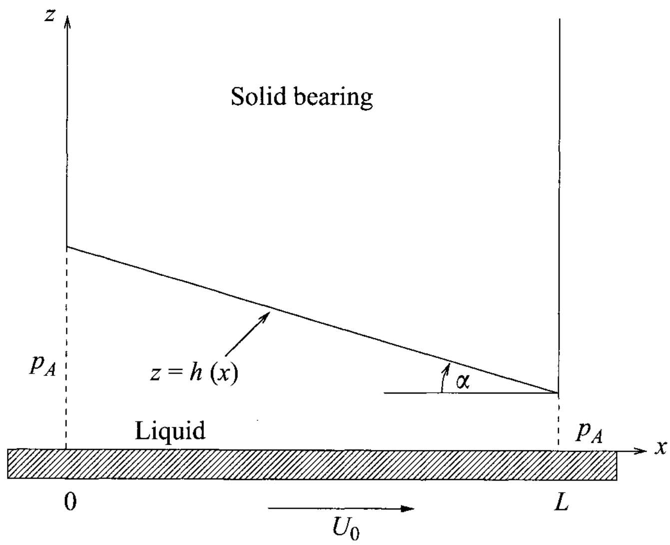

Figure 6. A sketch of a bearing supported by liquid dragged into the gap by the movement of the base plate.

which balances the jump in tangential stress with the surface-tension gradient: the Maragoni balance. For example if σ depends on temperature, then equation (2.5*d*) represents a balance between shear stress and surface-tension gradients. More complex rheological models for interfaces are discussed in Scriven (1960) and Slattery (1990).

In discussions to follow, if generalizations or augmentations of the above interfacial conditions are required, they will be determined *in situ*.

3 Thin films

Film dynamics and instability are central to a vast number of processes, as indicated in the Introduction. These will be analysed here by taking advantage of the geometrical disparity inherent in long-wave structures by using lubrication approximations.

3.1 Lubrication theory

The prototype problem in mechanics displaying geometrical disparity is the fluid-lubricated bearing as shown in figure 6. Here a heavy solid is supported by hydrodynamic forces when the fluid is forced beneath a slightly tilted surface and an underlying plate that drags the fluid into the gap; see

Schlichting (1968) for a discussion. If, as shown in figure 6, the upper boundary of the gap has the equation $z = h(x)$, then for a small tilt $\alpha = |\mathrm{d}h/\mathrm{d}x|$ must be small.

Assume that the flow is locally parallel so that, approximately, $\mu u_{zz} = p_x$. Here μ is the viscosity of the fluid, (u, w) is the velocity vector $\boldsymbol{v}$ in this two-dimensional example, and p is the pressure. Subscripts denote partial differentiation. If p_x were constant, this would represent parallel flow. When p_x varies, the profiles change with x.

The normal component of the Navier–Stokes equation is given approximately by $p_z = 0$ which makes the pressure including buoyancy independent of height. Finally, there is the continuity condition $u_x + w_z = 0$.

The boundary conditions on the plate and bearing are standard: for $0 < x < L$ (under the bearing) $u(x,0) = U_0$, $w(x,0) = 0$ and $u(x,h) = w(x,h) = 0$. In this approximate model realistic conditions at $x = 0$ and L are complicated, and instead of these one usually poses $p(0,z) = p(L,z) = p_A$, the atmospheric pressure.

Given that p depends on x only, one can integrate the first equation twice and use the conditions at the top and bottom to obtain the velocity profile

$$\mu u(x,z) = \tfrac{1}{2} p_x \left(z^2 - hz\right) + \mu U_0 \left(1 - \frac{z}{h}\right). \tag{3.1}$$

The flow is a linear combination of plane Couette flow (driven by the motion of the plate) and plane Poiseuille flow (driven by the pressure gradient p_x) induced by requiring that p have the same constant value at the exit and entrance.

Given that the flow is steady, the constant flow rate (in the x-direction) is given by $Q = \int_0^{h(x)} u(x,z)\mathrm{d}z$, and using equation (3.1)

$$\mu Q = -\tfrac{1}{12} h^3 p_x + \tfrac{1}{2} \mu U_0 h. \tag{3.2}$$

Alternatively, equation (3.2) can be differentiated to give

$$\left(-\tfrac{1}{12} h^3 p_x + \tfrac{1}{2} \mu U_0 h\right)_x = 0. \tag{3.3}$$

This is called the *Reynolds lubrication equation*. Given $h(x)$, it is an ordinary differential equation for p, subject to the given pressure conditions. Using either equation (3.2) or (3.3), one finds that

$$p(x) - p_A = 6\mu U_0 \left[\int_0^x h^{-2}\mathrm{d}x - \frac{\int_0^L h^{-2}\mathrm{d}x}{\int_0^L h^{-3}\mathrm{d}x} \int_0^x h^{-3}\mathrm{d}x\right]. \tag{3.4}$$

The maximum pressure $p_m \sim \alpha^{-1}$ and so the upward force F_v exerted by

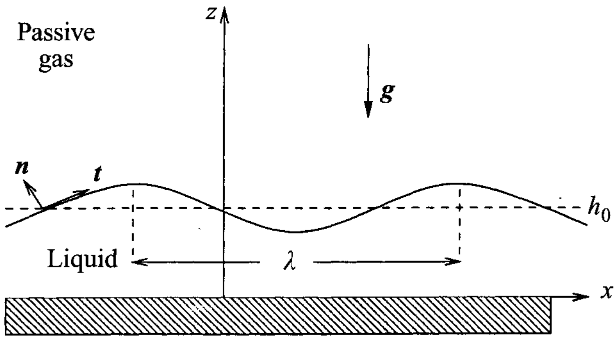

Figure 7. A sketch of a liquid film of mean thickness h_0 on a substrate. When the interface is disturbed (periodically in space) the wavelength $\lambda \gg h_0$. Gravity $\boldsymbol{g}$ acts vertically downward.

the fluid on the bearing scales as $F_v \sim \alpha^{-2}$, both as $\alpha \to 0$; a large weight can be supported by a fluid film. Such results can be generalized to three-dimensional and unsteady systems (see e.g. Oron, Davis & Bankoff 1997).

3.2 Liquid films on solid substrates

Consider a two-dimensional thin viscous film on a horizontal substrate as shown in figure 7. The mean thickness is h_0, the magnitude of the acceleration due to gravity is g, and the liquid has viscosity μ and density ρ, both constants. Further, the interface between the liquid and passive gas possesses a constant surface tension σ. The governing equations are Navier–Stokes

$$\rho(u_t + uu_x + wu_z) = -p_x + \mu(u_{xx} + u_{zz}), \tag{3.5a}$$

$$\rho(w_t + uw_x + ww_z) = -p_z + \mu(w_{xx} + w_{zz}) - \rho g, \tag{3.5b}$$

and continuity

$$u_x + w_z = 0. \tag{3.5c}$$

On the substrate, $z = 0$,

$$u = w = 0. \tag{3.5d}$$

On the interface, $z = h(x, t)$, there is the kinematic condition

$$w = h_t + uh_x, \tag{3.5e}$$

the condition of zero shear stress

$$(u_z + w_x)\left(1 - h_x^2\right) - 4h_x u_x = 0, \tag{3.5f}$$