LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES

Managing Editor: Professor N.J. Hitchin, Mathematical Institute,
University of Oxford, 24–29 St Giles, Oxford OX1 3LB, United Kingdom

The titles below are available from booksellers, or, in case of difficulty, from Cambridge University Press at www.cambridge.org.

113 Lectures on the asymptotic theory of ideals, D. REES
116 Representations of algebras, P.J. WEBB (ed)
119 Triangulated categories in the representation theory of finite-dimensional algebras, D. HAPPEL
121 Proceedings of *Groups - St Andrews 1985*, E. ROBERTSON & C. CAMPBELL (eds)
128 Descriptive set theory and the structure of sets of uniqueness, A.S. KECHRIS & A. LOUVEAU
130 Model theory and modules, M. PREST
131 Algebraic, extremal & metric combinatorics, M.-M. DEZA, P. FRANKL & I.G. ROSENBERG (eds)
138 Analysis at Urbana, II, E. BERKSON, T. PECK, & J. UHL (eds)
139 Advances in homotopy theory, S. SALAMON, B. STEER & W. SUTHERLAND (eds)
140 Geometric aspects of Banach spaces, E.M. PEINADOR & A. RODES (eds)
141 Surveys in combinatorics 1989, J. SIEMONS (ed)
144 Introduction to uniform spaces, I.M. JAMES
146 Cohen-Macaulay modules over Cohen-Macaulay rings, Y. YOSHINO
148 Helices and vector bundles, A.N. RUDAKOV *et al*
149 Solitons, nonlinear evolution equations and inverse scattering, M. ABLOWITZ & P. CLARKSON
150 Geometry of low-dimensional manifolds 1, S. DONALDSON & C.B. THOMAS (eds)
151 Geometry of low-dimensional manifolds 2, S. DONALDSON & C.B. THOMAS (eds)
152 Oligomorphic permutation groups, P. CAMERON
153 L-functions and arithmetic, J. COATES & M.J. TAYLOR (eds)
155 Classification theories of polarized varieties, TAKAO FUJITA
158 Geometry of Banach spaces, P.F.X. MÜLLER & W. SCHACHERMAYER (eds)
159 Groups St Andrews 1989 volume 1, C.M. CAMPBELL & E.F. ROBERTSON (eds)
160 Groups St Andrews 1989 volume 2, C.M. CAMPBELL & E.F. ROBERTSON (eds)
161 Lectures on block theory, BURKHARD KÜLSHAMMER
163 Topics in varieties of group representations, S.M. VOVSI
164 Quasi-symmetric designs, M.S. SHRIKANDE & S.S. SANE
166 Surveys in combinatorics, 1991, A.D. KEEDWELL (ed)
168 Representations of algebras, H. TACHIKAWA & S. BRENNER (eds)
169 Boolean function complexity, M.S. PATERSON (ed)
170 Manifolds with singularities and the Adams-Novikov spectral sequence, B. BOTVINNIK
171 Squares, A.R. RAJWADE
172 Algebraic varieties, GEORGE R. KEMPF
173 Discrete groups and geometry, W.J. HARVEY & C. MACLACHLAN (eds)
174 Lectures on mechanics, J.E. MARSDEN
175 Adams memorial symposium on algebraic topology 1, N. RAY & G. WALKER (eds)
176 Adams memorial symposium on algebraic topology 2, N. RAY & G. WALKER (eds)
177 Applications of categories in computer science, M. FOURMAN, P. JOHNSTONE & A. PITTS (eds)
178 Lower K- and L-theory, A. RANICKI
179 Complex projective geometry, G. ELLINGSRUD *et al*
180 Lectures on ergodic theory and Pesin theory on compact manifolds, M. POLLICOTT
181 Geometric group theory I, G.A. NIBLO & M.A. ROLLER (eds)
182 Geometric group theory II, G.A. NIBLO & M.A. ROLLER (eds)
183 Shintani zeta functions, A. YUKIE
184 Arithmetical functions, W. SCHWARZ & J. SPILKER
185 Representations of solvable groups, O. MANZ & T.R. WOLF
186 Complexity: knots, colourings and counting, D.J.A. WELSH
187 Surveys in combinatorics, 1993, K. WALKER (ed)
188 Local analysis for the odd order theorem, H. BENDER & G. GLAUBERMAN
189 Locally presentable and accessible categories, J. ADAMEK & J. ROSICKY
190 Polynomial invariants of finite groups, D.J. BENSON
191 Finite geometry and combinatorics, F. DE CLERCK *et al*
192 Symplectic geometry, D. SALAMON (ed)
194 Independent random variables and rearrangement invariant spaces, M. BRAVERMAN
195 Arithmetic of blowup algebras, WOLMER VASCONCELOS
196 Microlocal analysis for differential operators, A. GRIGIS & J. SJÖSTRAND
197 Two-dimensional homotopy and combinatorial group theory, C. HOG-ANGELONI *et al*
198 The algebraic characterization of geometric 4-manifolds, J.A. HILLMAN
199 Invariant potential theory in the unit ball of C^n, MANFRED STOLL
200 The Grothendieck theory of dessins d'enfant, L. SCHNEPS (ed)
201 Singularities, JEAN-PAUL BRASSELET (ed)
202 The technique of pseudodifferential operators, H.O. CORDES
203 Hochschild cohomology of von Neumann algebras, A. SINCLAIR & R. SMITH
204 Combinatorial and geometric group theory, A.J. DUNCAN, N.D. GILBERT & J. HOWIE (eds)
205 Ergodic theory and its connections with harmonic analysis, K. PETERSEN & I. SALAMA (eds)
207 Groups of Lie type and their geometries, W.M. KANTOR & L. DI MARTINO (eds)
208 Vector bundles in algebraic geometry, N.J. HITCHIN, P. NEWSTEAD & W.M. OXBURY (eds)
209 Arithmetic of diagonal hypersurfaces over infite fields, F.Q. GOUVÊA & N. YUI
210 Hilbert C^*-modules, E.C. LANCE
211 Groups 93 Galway / St Andrews I, C.M. CAMPBELL *et al* (eds)
212 Groups 93 Galway / St Andrews II, C.M. CAMPBELL *et al* (eds)
214 Generalised Euler-Jacobi inversion formula and asymptotics beyond all orders, V. KOWALENKO *et al*
215 Number theory 1992–93, S. DAVID (ed)

216 Stochastic partial differential equations, A. ETHERIDGE (ed)
217 Quadratic forms with applications to algebraic geometry and topology, A. PFISTER
218 Surveys in combinatorics, 1995, PETER ROWLINSON (ed)
220 Algebraic set theory, A. JOYAL & I. MOERDIJK
221 Harmonic approximation, S.J. GARDINER
222 Advances in linear logic, J.-Y. GIRARD, Y. LAFONT & L. REGNIER (eds)
223 Analytic semigroups and semilinear initial boundary value problems, KAZUAKI TAIRA
224 Computability, enumerability, unsolvability, S.B. COOPER, T.A. SLAMAN & S.S. WAINER (eds)
225 A mathematical introduction to string theory, S. ALBEVERIO, J. JOST, S. PAYCHA, S. SCARLATTI
226 Novikov conjectures, index theorems and rigidity I, S. FERRY, A. RANICKI & J. ROSENBERG (eds)
227 Novikov conjectures, index theorems and rigidity II, S. FERRY, A. RANICKI & J. ROSENBERG (eds)
228 Ergodic theory of $\mathbf{Z}^d$ actions, M. POLLICOTT & K. SCHMIDT (eds)
229 Ergodicity for infinite dimensional systems, G. DA PRATO & J. ZABCZYK
230 Prolegomena to a middlebrow arithmetic of curves of genus 2, J.W.S. CASSELS & E.V. FLYNN
231 Semigroup theory and its applications, K.H. HOFMANN & M.W. MISLOVE (eds)
232 The descriptive set theory of Polish group actions, H. BECKER & A.S. KECHRIS
233 Finite fields and applications, S.COHEN & H. NIEDERREITER (eds)
234 Introduction to subfactors, V. JONES & V.S. SUNDER
235 Number theory 1993–94, S. DAVID (ed)
236 The James forest, H. FETTER & B. GAMBOA DE BUEN
237 Sieve methods, exponential sums, and their applications in number theory, G.R.H. GREAVES *et al*
238 Representation theory and algebraic geometry, A. MARTSINKOVSKY & G. TODOROV (eds)
239 Clifford algebras and spinors, P. LOUNESTO
240 Stable groups, FRANK O. WAGNER
241 Surveys in combinatorics, 1997, R.A. BAILEY (ed)
242 Geometric Galois actions I, L. SCHNEPS & P. LOCHAK (eds)
243 Geometric Galois actions II, L. SCHNEPS & P. LOCHAK (eds)
244 Model theory of groups and automorphism groups, D. EVANS (ed)
245 Geometry, combinatorial designs and related structures, J.W.P. HIRSCHFELD *et al*
246 p-Automorphisms of finite p-groups, E.I. KHUKHRO
247 Analytic number theory, Y. MOTOHASHI (ed)
248 Tame topology and o-minimal structures, LOU VAN DEN DRIES
249 The atlas of finite groups: ten years on, ROBERT CURTIS & ROBERT WILSON (eds)
250 Characters and blocks of finite groups, G. NAVARRO
251 Gröbner bases and applications, B. BUCHBERGER & F. WINKLER (eds)
252 Geometry and cohomology in group theory, P. KROPHOLLER, G. NIBLO, R. STÖHR (eds)
253 The q-Schur algebra, S. DONKIN
254 Galois representations in arithmetic algebraic geometry, A.J. SCHOLL & R.L. TAYLOR (eds)
255 Symmetries and integrability of difference equations, P.A. CLARKSON & F.W. NIJHOFF (eds)
256 Aspects of Galois theory, HELMUT VÖLKLEIN *et al*
257 An introduction to noncommutative differential geometry and its physical applications 2ed, J. MADORE
258 Sets and proofs, S.B. COOPER & J. TRUSS (eds)
259 Models and computability, S.B. COOPER & J. TRUSS (eds)
260 Groups St Andrews 1997 in Bath, I, C.M. CAMPBELL *et al*
261 Groups St Andrews 1997 in Bath, II, C.M. CAMPBELL *et al*
263 Singularity theory, BILL BRUCE & DAVID MOND (eds)
264 New trends in algebraic geometry, K. HULEK, F. CATANESE, C. PETERS & M. REID (eds)
265 Elliptic curves in cryptography, I. BLAKE, G. SEROUSSI & N. SMART
267 Surveys in combinatorics, 1999, J.D. LAMB & D.A. PREECE (eds)
268 Spectral asymptotics in the semi-classical limit, M. DIMASSI & J. SJÖSTRAND
269 Ergodic theory and topological dynamics, M.B. BEKKA & M. MAYER
270 Analysis on Lie Groups, N.T. VAROPOULOS & S. MUSTAPHA
271 Singular perturbations of differential operators, S. ALBEVERIO & P. KURASOV
272 Character theory for the odd order function, T. PETERFALVI
273 Spectral theory and geometry, E.B. DAVIES & Y. SAFAROV (eds)
274 The Mandelbrot set, theme and variations, TAN LEI (ed)
275 Computational and geometric aspects of modern algebra, M. D. ATKINSON *et al* (eds)
276 Singularities of plane curves, E. CASAS-ALVERO
277 Descriptive set theory and dynamical systems, M. FOREMAN *et al* (eds)
278 Global attractors in abstract parabolic problems, J.W. CHOLEWA & T. DLOTKO
279 Topics in symbolic dynamics and applications, F. BLANCHARD, A. MAASS & A. NOGUEIRA (eds)
280 Characters and Automorphism Groups of Compact Riemann Surfaces, T. BREUER
281 Explicit birational geometry of 3-folds, ALESSIO CORTI & MILES REID (eds)
282 Auslander-Buchweitz approximations of equivariant modules, M. HASHIMOTO
283 Nonlinear elasticity, R. OGDEN & Y. FU (eds)
284 Foundations of computational mathematics, R. DEVORE, A. ISERLES & E. SULI (eds)
285 Rational points on curves over finite fields: Theory and Applications, H. NIEDERREITER & C. XING
286 Clifford algebras and spinors 2nd edn, P. LOUNESTO
287 Topics on Riemann surfaces and Fuchsian groups, E. BUJALANCE, A. F. COSTA & E. MARTINEZ (eds)
288 Surveys in combinatorics, 2001, J. W. P. HIRSCHFELD (ed)
289 Aspects of Sobolev-type inquealities, L. SALOFF-COSTE
290 Quantum groups and Lie Theory, A. PRESSLEY
291 Tits buildings and the model theory of groups, K. TENT
292 A quantum groups primer, S. MAJID
293 Second order partial differential equations in Hilbert spaces,. . .G. Da Prato & J. Zabczyk

The homotopy category of simply connected 4-manifolds

Hans-Joachim Baues

With an Appendix "On the cohomology of the category **nil**" by Teimuraz Pirashvili.

CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of
education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9780521531030

First published 2003

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-53103-0 Paperback

CONTENTS

INTRODUCTION

We study homotopy classes of maps between simply connected closed topological 4-manifolds. By a well known result of Freedman [F] each simply connected 4-dimensional Poincaré complex X is homotopy equivalent to such a manifold. Here X is a CW-complex

$$X = S^2 \vee \cdots \vee S^2 \cup_g e^4$$

obtained by attaching a 4-cell to a one point union of 2-dimensional spheres. The attaching map g is given by a unimodular form on the free abelian group $A = H_2X$. We call X a (2,4)-Poincaré complex and we consider the homotopy category $\mathbf{P}(2,4)$ consisting of (2,4)-Poincaré complexes and homotopy classes of maps. Then $\mathbf{P}(2,4)$ is equivalent to the homotopy category of simply connected 4-manifolds.

It is an old result that the homotopy type of such a manifold or Poincaré complex X is completely determined by the unimodular form associated to X; compare J.H.C. Whitehead [W4] and Milnor [MI]. The homotopy classes of maps between such Poincaré complexes, however, do not coincide with the homomorphisms between the associated forms. More precisely we obtain for a map $F : Y \to X$ in $\mathbf{P}(2,4)$ the induced maps in homology

$$\begin{aligned} \eta &= F_* : H_2Y \to H_2X, \\ \xi &= F_* : H_4Y = \mathbb{Z} \to H_4X = \mathbb{Z}, \end{aligned}$$

where $\xi \in \mathbb{Z}$ is the degree of F. The pair (ξ, η) is compatible with the unimodular forms f of Y and g of X and we call $(\xi, \eta) : f \to g$ a morphism between unimodular forms. Let **UF** be the category consisting of unimodular forms and such morphisms. Then homology yields a functor

$$H_* : \mathbf{P}(2,4) \to \mathbf{UF}$$

which is full, is representative and reflects isomorphisms. But this functor is not an equivalence of categories.

In fact, let $[Y,X]$ be the set of homotopy classes of maps $Y \to X$ and let

$$[Y,X]_{\xi,\eta} \subset [Y,X]$$

be the subset of all elements $F \in [Y,X]$ which induce $(\xi,\eta) = H_*F$ in homology. If ξ is non-trivial then $[Y,X]_{\xi,\eta}$ is a finite set. We compute an abelian group $D(\xi,\eta)$ which acts transitively and effectively on the set $[Y,X]_{\xi,\eta}$.

For example let

$$Aut(X) \subset [X,X]$$

be the group of homotopy equivalences of X and let $Aut(g)$ be the group of automorphisms of the form g of X in the category **UF**. Then homology H_* yields an extension of groups

$$D(1,1) \rightarrowtail Aut(X) \xrightarrow{H_*} \!\!\!\!\rightarrow Aut(g).$$

Here H_* is a surjective homomorphism and the kernel of H_* is given by the $(\mathbb{Z}/2)$-vector space

$$D(1,1) \; = \; ker(w : H_2(X,\mathbb{Z}/2) \to \mathbb{Z}/2)$$

defined by the Stiefel-Whitney class w of X. Moreover $D(1,1)$ is an $Aut(g)$-module so that the extension $Aut(X)$ is determined by a cohomology class

$$\{Aut(X)\} \in H^2(Aut(g), D(1,1)).$$

The class $\{Aut(X)\}$ is known to be trivial for all X in $\mathbf{P}(2,4)$ so that there is a homomorphism $Aut(g) \to Aut(X)$ of groups which splits the homomorphism $H_* : Aut(X) \to Aut(g)$.

The extension $Aut(X)$ above, in fact, admits a generalization on the level of categories. The transitive action of $D(\xi,\eta)$ on the set $[Y,X]_{\xi,\eta}$ (denoted by $+$) yields a linear extension of categories

$$D \xrightarrow{+} \mathbf{P}(2,4) \xrightarrow{H_*} \mathbf{UF}$$

which represents a cohomology class

$$\{\mathbf{P}(2,4)\} \in H^2(\mathbf{UF}, D).$$

Here we use the cohomology of categories with coefficients in natural systems introduced in [BW]. The cohomology class determines the extension $\mathbf{P}(2,4)$ up to equivalence. We prove that the cohomolgy class $\{\mathbf{P}(2,4)\}$ is a non trivial element. This implies that there is no functor $s : \mathbf{UF} \to \mathbf{P}(2,4)$ which splits the homology functor $H_* : \mathbf{P}(2,4) \to \mathbf{UF}$. This is a somewhat surprising result since the extension $Aut(X)$ above is known to be split for all X in $\mathbf{P}(2,4)$. Moreover we compute the subextension

$$D \longrightarrow \mathbf{P}(2,4)_0 \longrightarrow \mathbf{UF}_0$$

consisting of morphisms with non trivial degree. We determine the cohomology class $\{\mathbf{P}(2,4)_0\}$ explicitly which is a non trivial element of order 2. For this we describe three algebraic extensions

$$\begin{array}{ccccccccc} & & & & L & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathbf{L}(2,4)_0 & & & & \\ & & & & \downarrow & & & & \\ K^{st} & \longrightarrow & \mathbf{S}(2,4)_0 & \longrightarrow & \mathbf{UF}_0 & \longleftarrow & \mathbf{S}(odd) & \longleftarrow & K^u \end{array}$$

such that $\mathbf{P}(2,4)_0$ is equivalent to the pull back of these three extensions. Compare theorem (2.7.3). This is a complete algebraic characterization of the category $\mathbf{P}(2,4)_0$. As an application this yields the computation of the group $Aut(X)$ of homotopy equivalences of X originally obtained by Cochran and Habegger [CH], see section 2.5.

The category $\mathbf{S}(2,4)_0$ is the *stable part* of $\mathbf{P}(2,4)_0$, that is,

$$\mathbf{S}(2,4)_0 \ = \ \mathbf{P}(2,4)_0 / \overset{S}{\simeq}$$

is a quotient category of $\mathbf{P}(2,4)_0$ given by the stable homotopy relation $\overset{S}{\simeq}$. Here we have $F \overset{S}{\simeq} G$ for $F, G : Y \to X$ if the double suspensions $\Sigma^2 F, \Sigma^2 G$ are homotopic.

Next the category $\mathbf{L}(2,4)$ is the *Lie part* of $\mathbf{P}(2,4)_0$. The computation of the Lie part heavily relies on the methods in the book [BCU]. Let **nil** be the subcategory of groups consisting of groups $F/\Gamma_3 F$ where F is a finitely generated free group and $\Gamma_3 F$ is the subgroup of triple commutators. Then $\mathbf{L}(2,4)_0$ is algebraically derived from the category **nil**.

Finally the *odd part* $\mathbf{S}(odd)$ of $\mathbf{P}(2,4)_0$ is determined by the "signature derivation". The computation of $\mathbf{S}(odd)$ relies on a result of Rochlin [R] concerning the role of $\mathbb{Z}/16$ for simply connected 4-manifolds.

We also study the homotopy category $\mathbf{CW}(2,4)$ of (2,4)-complexes which are CW-complexes X of the form

$$X = S^2 \vee \cdots \vee S^2 \cup e^4 \cup \cdots \cup e^4$$

obtained by attaching a collection of 4-cells to a one-point union of 2-spheres. Of course (2,4)-Poincaré complexes are special (2,4)-complexes so that we have the inclusion of categories

$$\mathbf{P}(2,4) \subset \mathbf{CW}(2,4).$$

In chapter 1 we show that $\mathbf{CW}(2,4)$ is also part of a linear extension of categories

$$D \longrightarrow \mathbf{CW}(2,4) \xrightarrow{H_*} \mathbf{H}(2,4)$$

where H_* is the homology functor.

As a further main result in this book we compute an algebraic category $\mathbf{T}$ together with a natural equivalence relation $\simeq$ such that the quotient category $\mathbf{T}/\simeq$ admits an equivalence

$$\mathbf{T}/\simeq \; = \; \mathbf{CW}(2,4)$$

of categories which is also an equivalence of linear extensions. Compare chapter 5. The algebraic model $\mathbf{T}/\simeq$ of the category $\mathbf{CW}(2,4)$ relies on the construction of certain quadratic refinements of J.H.C. Whitehead's quadratic functor Γ in [W].

The computation of the algebraic category $\mathbf{T}/\simeq$, equivalent to $\mathbf{CW}(2,4)$, is an application of the theory on crossed modules and quadratic modules in the book [BCU]. The category $\mathbf{T}/\simeq$ yields a sophisticated algebraic description of the categories

$$\mathbf{P}(2,4)_0 \ \subset \ \mathbf{CW}(2,4) \ = \ \mathbf{T}/\simeq .$$

In fact, this is needed in order to compute the Lie part $\mathbf{L}(2,4)_0$ of $\mathbf{P}(2,4)_0$ above. We, however, do not achieve our result on the odd part $\mathbf{S}(odd)$ of $\mathbf{P}(2,4)_0$ by use of the category $\mathbf{T}$. Such a computation (independent of Rochlin's result [R]) remains an open question.

There has been great interest in the literature in constructing algebraic model categories of homotopy theory, in particular of rational homotopy theory. Over the integers $\mathbb{Z}$, however, there are only a very few explicit computations of homotopy categories in the literature. Our computation of $\mathbf{P}(2,4)_0$ and $\mathbf{CW}(2,4)$ achieves such algebraic characterization of homotopy categories over $\mathbb{Z}$.

Bonn, January 2001

Hans-Joachim Baues