Cambridge University Press & Assessment
978-0-521-53103-0 — The Homotopy Category of Simply Connected 4-Manifolds
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Excerpt
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CHAPTER 1

THE HOMOTOPY CATEGORY OF $(2,4)$-COMPLEXES

A (2,4)-complex is a CW-complex X with cells only in dimension 2 and 4. Hence X is of the form

$$X = S^2 \vee \cdots \vee S^2 \cup e^4 \cup \cdots \cup e^4$$

where we attach 4-cells to a one point union of 2-spheres. It is well known that a simply connected 4-manifold M is homotopy equivalent to a (2,4)-complex with only one 4-cell. Moreover if N is a simply connected 6-manifold with torsion free homology and $H_3N = 0$ then the 4-skeleton of N is a (2,4)-complex. For example the 4-skeleton of the product $N = S^2 \times S^2 \times S^2$ is the (2,4)-complex

$$N^4 = S^2 \times S^2 \times * \cup S^2 \times * \times S^2 \cup * \times S^2 \times S^2.$$

In this chapter we consider the homotopy category $\mathbf{CW}(2,4)$ of (2,4)-complexes and we show that $\mathbf{CW}(2,4)$ is part of a linear extension of categories. In chapter 5 we will describe an algebraic category which is equivalent to $\mathbf{CW}(2,4)$.

1.1. Quadratic functions and the Hopf map

We recall from the literature some basic facts on low dimensional homotopy groups π_3 and π_4. J.H.C. Whitehead [W] showed that the Hopf map $\eta : S^3 \to S^2$ has a quadratic distributivity law. Therefore the quadratic functor Γ can be used to describe the homotopy groups π_3 and π_4 of a Moore space $M(A, 2)$ where A is a free abelian group.

A function $f : A \to B$ between abelian groups is called *quadratic* if $f(-a) = f(a)$ and if the function $A \times A \to B$, given by $(a, b) \mapsto f(a+b) - f(a) - f(b)$, is bilinear. There is the universal quadratic map

(1.1.1) $$\gamma : A \longrightarrow \Gamma(A)$$

with the property: for all quadratic maps $f : A \to B$ there is a unique (induced) homomorphism $f^{\square} : \Gamma(A) \to B$ with $f^{\square}\gamma = f$. A homomorphism $\varphi : A' \to A$ yields the quadratic map $\gamma\varphi$ which induces $\Gamma(\varphi) = (\gamma\varphi)^{\square} : \Gamma(A') \to \Gamma(A)$.

This shows that $\Gamma : \mathbf{Ab} \to \mathbf{Ab}$ is a well defined functor, where $\mathbf{Ab}$ denotes the category of abelian groups. We associate with Γ the following natural commutative diagram in which the subdiagram "push" is a push out of abelian groups; moreover $\otimes^2 A = A \otimes A$ denotes the *tensor product* over $\mathbb{Z}$.

(1.1.2) $$\begin{array}{ccccc} A \otimes A & & & & \\ \downarrow{\scriptstyle [1,1]} & & & & \\ \Gamma(A) & \xrightarrow{\tau} & A \otimes A & \xrightarrow{q} & A \wedge A \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \bar{\sigma}} & & \downarrow{\scriptstyle id} \\ A \otimes \mathbb{Z}/2 & \xrightarrow{\bar{\tau}} & A \hat{\otimes} A & \xrightarrow{q} & A \wedge A \end{array}$$

The column and the rows of the diagram are exact sequences of abelian groups. The homomorphism τ is induced by the quadratic map $A \to A \otimes A$, $A \mapsto a \otimes a$, and σ is induced by $a \mapsto a \otimes 1$. Here $1 \in \mathbb{Z}/2$ ist the generator of $\mathbb{Z}/2 = \mathbb{Z}/2\mathbb{Z}$. We define $[1, 1]$ by

(1) $$[1, 1](a \otimes b) = [a, b] = \gamma(a + b) - \gamma(a) - \gamma(b).$$

We clearly have $[a, b] = [b, a]$, $[a, a] = 2\gamma(a)$, $\Gamma(\varphi)[a, b] = [\varphi a, \varphi b]$, $\sigma[a, b] = 0$ and $\tau[a, b] = a \otimes b + b \otimes a$.

We obtain the *exterior product* $\Lambda^2 A = A \wedge A = A \otimes A/\sim$ by the relation $\tau\gamma(a) = a \otimes a \sim 0$ and we obtain $\hat{\otimes}^2 A = A \hat{\otimes} A = A \otimes A/\sim$ by the relation $\tau[a, b] = a \otimes b + b \otimes a \sim 0$. The quotient map q carries $a \otimes b$ to $a \wedge b$.

If A is free abelian with ordered basis Z, then ΓA and $A \wedge A$ are free abelian with the basis $\{\gamma(m), [m,n] : m < n,\ m,n \in Z\}$ and $\{m \wedge n : m < n,\ m,n \in Z\}$ respectively. In this case the homomorphisms τ ans $\bar{\tau}$ in (1.1.2) are injective. If A is free abelian we define the (integral) *matrix* $(x_{m,n} : m,n \in Z)$ *of an element* $x \in \Gamma A$ by the formula

$$(2) \qquad \tau(x) \;=\; \sum_{m,n \in Z} x_{m,n} m \otimes n.$$

We say that x is *unimodular* if Z is a finite set and if the matrix of x satisfies determinant$(x_{m,n}) \in \{1,-1\}$. The matrix $(x_{m,n})$ of x is a symmetric integral matrix since $x_{m,n} = x_{n,m}$ which determines the element x by

$$x \;=\; (\sum_{m<n} x_{m,n}[m,n]) + (\sum_{m} x_{m,m}\gamma(m)).$$

This way we identify symmetric integral matrices with elements in $\Gamma(A)$.

Recall that a *Moore space* $M(A,n),\ n \geq 2$, is a simply connected CW-complex with reduced homology groups $H_i M(A,n) = 0$ for $i \neq n$ and $H_n M(A,n) = A$. For a free abelian group A with basis Z we set

$$(1.1.3) \qquad M(A,n) \;=\; \bigvee_{Z} S^n$$

where the right hand side is a one point union of n-spheres over the index set Z. The element $m \in Z$ yields the inclusion $m : S^n \subset M(A,n)$ also denoted by m. This is compatible with the natural identification $A = H_n M(A,n) = \pi_n M(A,n)$. Recall that for pointed spaces X, Y we have the *set* $[X,Y]$ *of homotopy classes* of basepoint preserving maps $X \to Y$, for example $[S^n, Y] = \pi_n Y$ is the nth homotopy group of Y. For a free abelian group A we identify

$$(1) \qquad [M(A,n), Y] \;=\; Hom(A, \pi_n Y),$$

so that

$$(2) \qquad [M(A,n), M(B,n)] \;=\; Hom(A,B)$$

Here $Hom(A,B)$ is the abelian group of homomorphisms from A to B. The isomorphism (1) carries a map f to the induced homomorphism $\pi_n(f)$.

The *Hopf map* $\eta : S^3 \to S^2$ induces a quadratric map $\eta^* : \pi_2 Y \to \pi_3 Y$ satisfying the formula

$$\eta^*(a+b) \;=\; \eta^*(a) + \eta^*(b) + [a,b] \tag{1.1.4}$$

where $[a,b]$ is the Whitehead product. The quadratic map η^* induces a homomorphism

$$i = (\eta^*)^{\square} : \Gamma(\pi_2 Y) \to \pi_3 Y$$

between abelian groups. This homomorphism is part of Whitehead's certain exact sequence below.

Let Y be a connected CW-complex with skeletons Y^n and basepoint $* \in Y^0$ and let $\hat{Y}$ be the universal covering of Y. Then the Hurewicz homomorphism $(n \geq 2)$

$$h = h_Y : \pi_n Y \cong \pi_n \hat{Y} \to H_n \hat{Y}$$

is part of the long exact sequence

$$H_{n+1}\hat{Y} \xrightarrow{b} \Gamma_n Y \xrightarrow{i} \pi_n Y \xrightarrow{h} H_n \hat{Y} \xrightarrow{b} \Gamma_{n-1} Y$$

where the groups $\Gamma_n Y$ are defined by $\Gamma_n Y \;=\; im(\; \pi_n Y^{n-1} \xrightarrow{i_*} \pi_n Y^n\;)$ with i_* induced by the inclusion $i : Y^{n-1} \hookrightarrow Y^n$. One has a quadratic map $\eta^* : \pi_2 Y \to \Gamma_3 Y$ which carries $\alpha : S^2 \to Y^2 \subset Y^3$ to $\alpha\eta$ and the induced map

$$\Gamma(\pi_2 Y) \xrightarrow{\cong} \Gamma_3 Y$$

is an isomorphism for all connected spaces Y. Hence one has the exact sequence

$$\begin{array}{l} H_5\hat{Y} \xrightarrow{b} \Gamma_4 Y \xrightarrow{i} \pi_4 Y \xrightarrow{h} H_4\hat{Y} \xrightarrow{b} \\ \Gamma(\pi_2 Y) \xrightarrow{i} \pi_3 Y \xrightarrow{h} H_3 \hat{Y} \longrightarrow 0. \end{array} \tag{1.1.5}$$

The operator $b = b_Y$ is the secondary boundary operator of Whitehead, see

[W], [BAH], [BCH], [BHH]. As an application of (1.1.5) one gets the natural isomorphism

(1.1.6) $$i : \Gamma(A) = \pi_3 M(A, 2)$$

which we use as an identification. This way the map γ in (1.1.1) corresponds to η^* and $[1, 1]$ in (1.1.2) corresponds to the Whitehead product $\pi_2 \otimes \pi_2 \to \pi_3$. Moreover σ in (1.1.2) corresponds to the suspension homomorphism Σ,

(1.1.7) $$\sigma : \Gamma(A) = \pi_3 M(A, 2) \xrightarrow{\Sigma} \pi_4 M(A, 3) = A \otimes \mathbb{Z}/2.$$

We now consider the next homotopy group $\pi_4 M(A, 2)$. For this we need the following definition.

1.1.8. Definition: Let $T(A, 1)$ be the free graded tensor algebra generated by the abelian group A where A is concentated in degree 1, that is $T(A, 1)_n = \otimes^n A$ is the n-fold tensor product of A. We define the structure of a graded Lie algebra on $T(A, 1)$ by

(1) $$[x, y] \;=\; xy - (-1)^{|x||y|} yx$$

for $x, y \in T(A, 1)$. As usual we set $|x| = n$ if $x \in T(A, 1)_n$. Let $L(A, 1)$ be the sub Lie algebra generated by A in $T(A, 1)$ and let $L(A, 1)_n = L(A, 1) \cap \otimes^n A$. Clearly $L(\_, 1)_n$ is a functor $\mathbf{Ab} \to \mathbf{Ab}$. We obtain

(2) $$L(A, 1)_3 \;=\; image[[1, 1], 1] : \otimes^3 A \to \otimes^3 A$$

where $[[1, 1], 1]$ carries $a \otimes b \otimes c$ to $[[a, b], c] = (a \otimes b + b \otimes a) \otimes c - c \otimes (a \otimes b + b \otimes a)$. Hence we have the quotient map

(3) $$\otimes^3 A \twoheadrightarrow L(A, 1)_3$$

which carries $a \otimes b \otimes c$ to $[[a, b], c]$ The kernel of this map is generated by the following elements:

(a) $a \otimes b \otimes c + c \otimes a \otimes b + b \otimes c \otimes a$,
(b) $a \otimes b \otimes c - b \otimes a \otimes c$,
(c) $a \otimes a \otimes a$.

Compare 11.1.5 in [BHH]

1.1.9. Proposition: *Let A be a free abelian group. Then one has natural isomorphisms*

$$\begin{aligned}\pi_4 M(A,2) \cong \Gamma_2^2 A &= \Gamma(A)\otimes \mathbb{Z}/2 \oplus L(A,1)_3 \\ &\cong (\Gamma(A)\otimes\mathbb{Z}/2 \oplus \Gamma(A)\otimes A)/R(A).\end{aligned}$$

Here the natural subgroup $R(A)$ is generated by the following elements ($x,y,z \in A$).

(i) $[x,y]\otimes z \;+\; [z,x]\otimes y \;+\; [y,z]\otimes x,$
(ii) $(\gamma x)\otimes x,$
(iii) $[x,y]\otimes 1 \;+\; (\gamma x)\otimes y \;+\; [y,x]\otimes x.$

Proof: For $\pi_i = \pi_i M(A,2)$ we have the natural homomorphisms

$$(\Sigma\eta)^* : \Gamma(A)\otimes\mathbb{Z}/2 = \pi_3\otimes\mathbb{Z}/2 \longrightarrow \pi_4, \tag{1}$$

$$w : \Gamma(A)\otimes A = \pi_3\otimes\pi_2 \longrightarrow \pi_4 \tag{2}$$

which are defined by $\Sigma\eta : S^4 \to S^3$ and by the Whitehead product $w = [-,-]$ respectively. Moreover we define

$$[[1,1],1] : \otimes^3 A \xrightarrow{w'} \Gamma(A)\otimes A \longrightarrow \pi_4 \tag{3}$$

by the triple Whitehead product, that is $w' = [1,1]\otimes 1$. This map induces the inclusion $[[1,1],1] : L(A,1)_3 \subset \pi_4$ which together with (1) yields the first isomorphism in (1.1.9). The second isomorphism is induced by (1) and (2); this shows that (i) corresponds to the Jacobi identity for Whitehead products, (ii) corresponds to the identity $[\eta, i_2] = 0$ for generators $\eta \in \pi_3(S^2)$, $i_2 \in \pi_2(S^2)$ and (iii) corresponds to a special case of the Barcus-Barratt formula

$$[x\eta, y] = [x,y]\Sigma\eta - [[y,x],x]. \tag{4}$$

Using the Hilton Milnor theorem one can check that the maps in (1.1.9) are actually isomorphisms, compare [BCC]. The splitting for $L(A,1)_3$ is obtained by the James-Hopf invariant γ_3 for which the composition

(5) $$L(A,1)_3 \xrightarrow{[[1,1],1]} \pi_4 M(A,2) \xrightarrow{\gamma_3} \otimes^3 A$$

is the inclusion in (1.1.8)(2). Compare also 11.1.9 of [BHH] and [BG]. □

We now consider the double suspension

(1.1.10) $$\Sigma^2 : \pi_4 M(A,2) \longrightarrow \pi_6 M(A,4) = A \otimes \mathbb{Z}/2$$

which is trivial on $L(A,1)_3$ and $\Gamma A \otimes A$ and which is given by $\sigma \otimes \mathbb{Z}/2 : \Gamma(A) \otimes \mathbb{Z}/2 \to A \otimes \mathbb{Z}/2 \otimes \mathbb{Z}/2$ on $\Gamma(A) \otimes \mathbb{Z}/2$, see (1.1.7). As usual we use the notation $f \otimes A = f \otimes 1_A$ where $1_A = 1$ is the identity of A and where f is a homomorphism.

1.2. Simply connected 4-manifolds and (2, 4)-complexes

We say that a CW-complex X is a (2,4)-*complex* if $X^0 = *$ and if $X - *$ consists only of 2-cells and 4-cells. Then X is the mapping cone $X = C_g$ of an attaching map

(1.2.1) $$g : M(B,3) \longrightarrow M(A,2)$$

where $A = H_2X$ and $B = H_4X$ are free abelian groups, see (1.1.3). The homotopy class of g is a homomorphism

(1) $$b_X = g : B = H_4X \longrightarrow \Gamma(A) = \Gamma(H_2X)$$

via (1.1.3)(1) and (1.1.6). Here g coincides with the secondary boundary b_X in Whitehead's exact sequence (1.1.5). The *cup product* pairing $\cup$ for the cohomology $H^*(X)$ is obtained by the commutative diagram

(2) $$\begin{array}{ccc} H^2(X) \otimes H^2(X) & = & Hom(A,\mathbb{Z}) \otimes Hom(A,\mathbb{Z}) \\ \downarrow \cup & & \downarrow (\tau g)^* \\ H^4(X) & = & Hom(B,\mathbb{Z}) \end{array}$$

where $(\tau g)^*(\alpha \otimes \beta) = (\alpha \otimes \beta)\tau g$, see (1.1.2). This shows that the homotopy class of g in (1.2.1) is also determined by the cohomology ring $H^*(X)$.

We say that a (2,4)-complex X as above is a (2,4)-*Poincaré complex* if $H_4X = B = \mathbb{Z}$, with the *fundamental class* $[X] \in H_4X$ as a generator, and if the element

(3) $$b_X[X] \;=\; g(1) \in \Gamma(A) = \Gamma(H_2X)$$

is unimodular, see (1.1.2)(2). Here we assume A to be a finitely generated free abelian group with basis Z. We call $b_X[X]$ the *quadratic form* of X.

Let Z be a basis of A and Z^* be the dual basis of $H^2X = Hom(A, \mathbb{Z})$. Then the matrix $(g_{m,n})$ of $g(1) \in \Gamma(A)$ with respect to Z is also the matrix of the cup product pairing with respect to the dual basis Z^*, that is for $m^* \in Z^*$ corresponding to $m \in Z$ one gets

(4) $$(m^* \cup n^*)([X]) \;=\; g_{m,n}.$$

We have the *Poincaré duality isomorphism*

(5) $$\psi = \psi_g : H_2X \cong H^2X$$

which carries $z \in H_2X = A$ to the unique element $x \in H^2X = Hom(A, \mathbb{Z})$ with $(x \otimes A)\tau g(1) = z$. Here we use the composite

$$\Gamma(A) \xrightarrow{\;\tau\;} A \otimes A \xrightarrow{x \otimes A} \mathbb{Z} \otimes A = A.\cdot$$

The *intersection form* of X is the bilinear form

(6) $$\cap : H_2X \otimes H_2X \to \mathbb{Z}$$

which carries $a \otimes b$ to $(\psi a) \cup (\psi b)$. Then $\psi^{-1}Z^*$ is the *Poincaré dual basis* of Z in $A = H_2X$. The matrix of the intersecion form with respect to $\psi^{-1}Z^*$ is $(g_{m,n})$ by (4).

We consider the full inclusion of homotopy categories

(1.2.2) $$\mathbf{P}(2,4) \subset \mathbf{CW}(2,4) \subset \mathbf{Top}^*/\simeq .$$

Here **Top*** is the category of pointed topological spaces; the morphisms $X \to Y$ in the homotopy category **Top**$^*/\simeq$ are the elements of the homotopy set $[X, Y]$. The categories $\mathbf{CW}(2,4)$ and $\mathbf{P}(2,4)$ are the full subcategories of **Top**$^*/\simeq$ consisting of (2,4)-complexes and of (2,4)-Poincaré complexes respectively. By a result of Freedman [F] any (2,4)-Poincaré complex has the homotopy type of a 1-*connected* 4-*dimensional closed topological manifold* so that $\mathbf{P}(2,4)$ is equivalent to the homotopy category of such manifolds.

We now compare the maps in $\mathbf{CW}(2,4)$ with the induced maps in homology. For this we introduce the category $\mathbf{H}(2,4)$. The objects of this category are homomorphisms $g : B \to \Gamma(A)$ where A and B are free abelian groups. The morphisms $(\xi, \eta) : f \to g$ are given by commutative diagrams

(1)
$$\begin{array}{ccc} B' & \xrightarrow{\ \xi\ } & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \Gamma(A') & \xrightarrow[\Gamma(\eta)]{} & \Gamma(A) \end{array}$$

There is the *homology functor*

(2)
$$H_* : \mathbf{CW}(2,4) \to \mathbf{H}(2,4)$$

which carries a $(2,4)$-complex X to the homotopy class $g = b_X$ of its attaching map, that is

(3)
$$H_*(X) \ = \ \{b_X : H_4X \to \Gamma(H_2X)\}$$

is given by Whitehead's exact sequence. The naturality of this sequence shows that H_* is a well defined functor. The restriction of this functor to (2,4)-Poincaré complexes yields the functor

(4)
$$H_* : \ \mathbf{P}(2,4) \longrightarrow \mathbf{UF}$$

where **UF** is the full *subcategory of unimodular forms* in $\mathbf{H}(2,4)$. The objects of **UF** are homomorphisms $g : B = \mathbb{Z} \to \Gamma(A)$ for which $g(1)$ is unimodular, see (1.2.1)(3). A morphism $(\xi, \eta) : f \to g$ in **UF** yields an integer $\xi \in \mathbb{Z}$ which we call the *degree*. For a map $F : Y \to X$ between (2,4)-Poincaré complexes with $H_*(F) = (\xi, \eta)$ we have $(H_4F)[Y] = \xi[X]$ and $H_2F = \eta$. We point out that for a morphism (ξ, η) in **UF** the map η determines ξ. In fact, in the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\xi} & \mathbb{Z} \\ f\downarrow & & \downarrow g \\ \Gamma(A') & \xrightarrow[\Gamma(\eta)]{} & \Gamma(A) \end{array}$$

the maps f and g are injective so that there is a unique element $\xi \in \mathbb{Z}$ with

$$\Gamma(\eta)(f(1)) \;=\; \xi \cdot g(1). \tag{5}$$

Recall that a functor $F : \mathbf{C} \to \mathbf{K}$ is *full* if F is surjective on morphism sets. Moreover F is *faithful* if F is injective on morphism sets. The functor F is *representative* if for each object K in $\mathbf{K}$ there exists an object X in $\mathbf{C}$ together with an isomorphism $F(X) \cong K$ in $\mathbf{K}$. The functor F *reflects isomorphisms* if a map $f : X \to Y$ in $\mathbf{C}$ is an isomorphism in $\mathbf{C}$ if and only if the induced map $F(f) : F(X) \to F(Y)$ is an isomorphism in $\mathbf{K}$.

1.2.3. PROPOSITION: *The homology functor*

$$H_* : \mathbf{CW}(2,4) \to \mathbf{H}(2,4)$$

is full and representative and H_* reflects isomorphisms. The functor H_* is not faithfull.

The proposition implies that H_* induces a bijection between homotopy types of (2,4)-complexes and isomorphism classes of objects in $\mathbf{H}(2,4)$.

We now consider the homotopy groups π_3, π_4 which are functors

$$\pi_3, \pi_4 : \mathbf{CW}(2,4) \to \mathbf{Ab} \tag{1.2.4}$$

For a (2,4)-complex X these groups are part of the exact sequence

$$0 \longrightarrow \Gamma_4 X \longrightarrow \pi_4 X \longrightarrow H_4 X \xrightarrow{g} \Gamma(H_2 X) \longrightarrow \pi_3 X \longrightarrow 0.$$