London Mathematical Society Lecture Note Series. 295

Geometry and Integrability

Edited by

Lionel Mason
University of Oxford

and

Yavuz Nutku
Feza Gurzey Institute, Instanbul

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011–4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2003

Printed in the United Kingdom at the University Press, Cambridge

Typeface Computer Modem 11/13.5 pt. *System* LaTeX [TB]

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication data

Geometry and integrability / edited by L. J. Mason, Y. Nutku.
p. cm. – (London Mathematical Society lecture note series; v. 295)
Includes bibliographical references and index.
ISBN 0 521 52999 9
1. Global differential geometry. 2. Twistor theory. 3. Fiber spaces
(Mathematics) I. Mason, L. J. (Lionel J.) II. Nutku, Yavuz.
III. London Mathematical Society lecture note series, 295.
QA670.G463 2003
516.3′62 – dc21 2003048456

ISBN 0 521 52999 9 paperback

Contents

1

Introduction

Lionel Mason
The Mathematical Institute, 24-29 St Giles, Oxford OX1 3LB, UK

1.1 Background

Integrable systems are systems of partial or ordinary differential equations that combine nontrivial nonlinearity with unexpected tractability. Often one can find large families of exact solutions, and general methods for generic solutions. This volume is concerned with the deep links that integrability has with geometry. There are two rather different ways that geometry emerges in the study of integrable systems.

1.1.1 Geometrical context for integrable equations

The first is from the context of the differential equations themselves: even those integrable equations whose origins, perhaps in the theory of water waves or plasma physics, seem a long way from geometry can usually be expressed in the context of symplectic geometry as possibly infinite dimensional Hamiltonian systems with many conserved quantities and often with much more further structure. But geometry is itself also a rich source of integrable systems; one of the first examples of a completely integrable nonlinear partial differential equation, the sine-Gordon equation first appeared in the 19th century theory of surfaces, as a formulation of the constant mean curvature condition on a 2-surface embedded in Euclidean 3-space. Now there are many more examples from geometry in many dimensions, from the two-dimensional systems given by harmonic maps from Riemann surfaces to symmetric spaces, to the anti-self-duality equations in 4-dimensions and more generally quaternionic structures in $4k$-dimensions.

The contributions of Tod, Mason and Woodhouse focus on the anti-self-duality equations either on a Yang–Mills connection on a vector bundle over $\mathbb{R}^4$, or on a 4-dimensional conformal structure. The systems discussed by Santini also have a geometric origin, in their discrete form as quadrilateral lattices, and in their continuous limits as conjugate nets. The reductions and specializations of these systems then form many more geometrical examples of integrable systems: although the systems discussed by Donagi are presented as arising from complex algebraic geometry rather than Riemannian geometry, they have their origin in reductions of the real anti-self-dual Yang-Mills equations.

1.1.2 Geometrical transforms and solution methods for differential equations

The second way that geometry appears in the theory of integrable systems is in the transforms and solution methods that are brought to bear on integrable systems. There are many different strands here. The symplectic framework for integrable equations leads to the first definition of an integrable system, that due to Arnol'd and Liouville, in terms the existence of sufficiently many constants of motion satisfying various requirements. The Arnol'd–Liouville theorem leads to a transform of the system to action-angle variables by quadratures in which the action variables are constant and the motion is linear in the angle variables. In fact many interesting integrable systems admit further structures that imply Arnol'd–Liouville integrability. Those considered by Donagi are algebraically completely integrable so that the structures in question are complexified and required furthermore to be algebraic. Another structure that guarantees complete integrability is a bi-Hamiltonian structure.

These structures in finite dimensions lead, at least in principle, to the general solution by quadratures. Integrable partial differential equations can often be expressed as infinite dimensional examples of systems satisfying the Arnol'd–Liouville requirements often by virtue of admitting a bi-Hamiltonian structure. However, the infinite number of degrees of freedom mean that one can no longer solve the system in a finite number of quadratures. Nevertheless, new techniques become available. On the one hand there are hidden symmetries, both discrete, such as Backlund transforms, and continuous, such as those generated by flows associated to the Arnol'd–Liouville constants of motion, and these can help generate new exact solutions. But also there are transforms that apply to general solutions; historically, the inverse scattering transform was the first important example of this and was used to provide the transform to action angle variables for solutions subject to rapidly decreasing boundary conditions in precise analogy with the transform provided by the finite dimensional Arnol'd–Liouville theorem.

There are now a number of such transforms such as the inverse spectral transform, the Penrose and Ward transforms and so on. A remarkable feature of many of these transforms is the appearance of sophisticated complex holomorphic, and often even algebraic geometry. This complex analysis often plays a deep role in the finite dimensional case also. In the contribution of Woodhouse we see twistor theory as providing a similar transform between solutions to integrable equations and geometric structures, holomorphic vector bundles, that can be described in terms of free functions. This construction has the additional benefit that it applies to the general local analytic solution. A related method is based on the non-local $\bar{\partial}$-problem, so called $\bar{\partial}$-dressing. In the local case this is often simply an independent formulation of the twistor correspondence, but in the non-local case, such constructions go beyond standard twistor theory.

1.2 The contributions

The following is intended to provide some introduction to, and context for, the various contributions. I should make a disclaimer here that the context and background I give are perhaps rather one-sided and reflect my own point of view; there are a number of different points of view that might be taken on this material that are not presented here!

1.2.1 Notes on reductions of the anti-self-dual Yang-Mills equations and integrable systems, L. J. Mason; Curvature and integrability for Bianchi-type IX metrics, K. P. Tod; Twistor theory and integrability, N. M. J. Woodhouse

These contributions are connected by an overview on the theory of integrable systems based on reductions of the anti-self-dual Yang-Mills (ASDYM) equations and anti-self-dual conformal structures.

The ASDYM equations can be thought of as integrable by virtue of the existence of the Ward correspondence between solutions to these equations and holomorphic vector bundles on an auxilliary complex manifold, twistor space. For ASDYM fields on Minkowski space, twistor space is a portion of $\mathbb{CP}^3$, complex projective 3-space. If one allows the transform between solutions to the ASDYM equations and twistor data, this construction amounts to providing, in a geometric form, the general solution to the ASDYM equations. There is a similar construction due to Penrose giving a correspondence between anti-self-dual conformal structures and deformations of the complex structure on twistor space.

A key observation of Richard Ward's is that many of the most famous integrable equations are symmetry reductions of the ASDYM equations. The various aspects of the integrability of such reductions of the ASDYM equations can then be understood by reduction of the corresponding theory for the full ASDYM equations.

The contribution of Mason concerns the integrability of the ASDYM equations that can be understood without using twistor theory. Thus its Lax pair, Backlund transformations, Hamiltonian formulation and recursion operator and hierarchy are presented. Some of the more significant reductions are reviewed also.

Paul Tod's lectures on spinor calculus and conformal invariance were taken from his book with Huggett, Introduction to Twistor Theory (second edition), published by CUP as LMS Student-Text 5, and so are not included here. The book gives useful details of space-time geometry that provide a background for the twistor correspondence and the interested reader can refer to it for full details.

Nick Woodhouse's contribution is an introduction to twistor methods and explains how the Ward transform applies to ASDYM fields and descends to provide correspondences for reductions of ASDYM fields. In particular it is shown how twistor methods can give new insight into the KdV equations and the isomonodromy problem that arises in the study of Painlevé equations. One aspect of integrability that emerges particularly clearly is a 'geometric' explanation of the Painlevé test for integrability. The lectures build on those of Tod and Mason.

There is a further contribution from Paul Tod which concerns various equations on metrics in 4-dimensions that admit an SU(2) symmetry. The metric may be required to be Kahler, Einstein or have anti-self-dual Weyl tensor. The latter equation is usually thought to imply integrability because of Penrose's twistor correspondence. With this symmetry, the equations reduce to ODE's. If the metric is Einstein, it is no restriction to assume it is diagonal (although it is a nontrivial restriction for general anti-self-dual conformal structures). When Ricci flat, one obtains (with a further assumption) the Chazy equation. This is somewhat of a novelty for integrable systems theory as this equation admits solutions with movable natural boundaries, contradicting the Painlevé property. An explanation of this paradox is proposed.

1.2.2 Geometry and integrability, R.Y. Donagi

The contribution of Donagi is concerned with the theorem that the Moduli space of 'meromorphic Higgs bundles' over a Riemann surface Σ has the structure of an algebraically completely integrable system. This combines the symplectic geometry underlying the Arnol'd–Liouville definition of an integrable system with algebraic geometry. The Arnol'd–Liouville definition of a completely integrable system as above can be abstracted by taking an integrable system to mean a Poisson manifold, M, with sufficiently many commuting Hamiltonians, the collection being thought of as a map from $H : M \mapsto \mathbb{R}^n$, satisfying certain technical requirements to guarantee satisfactory global properties. This definition can be complexified so that M is a complex manifold and the Poisson structure is a complex holomorphic bivector and $H : M \mapsto \mathbb{C}^n$ are holomorphic. The algebraic condition is then that M be an algebraic manifold with all the structures being expressible in terms of algebraic functions of algebraic coordinates on M. Although this definition might seem somewhat special, there are a remarkable number of interesting systems that turn out to be integrable in this way.

A Higgs bundle E is a holomorphic vector bundle equipped with a global holomorphic section, the Higgs field Φ, of the associated bundle of 1-forms with values in the endomorphisms of E, $\mathrm{End}(E) \otimes \Omega^1(\Sigma)$. These first arose in the context of Hitchin's study of reductions of the anti-self-dual Yang-Mills equations on a connection on a bundle over Euclidean $\mathbb{R}^4$ by two translational symmetries. Remarkably, the reduced sytem acquires 2-dimensional conformal invariance and so makes sense on an arbitrary Riemann surface. The anti-self-duality condition reduces to equations on a connection and Higgs field on the Riemann surface; the Higgs field should be holomorphic and the curvature of the connection be given in terms of the Higgs fields. According to the philosophy of the contributions by Woodhouse, Tod and Mason, this Hitchin system is an integrable system. Since it is a system of elliptic partial differential equations, it doesn't naturally fall into a Hamiltonian framework. However, the space of solutions on a compact Riemann surface is finite dimensional and one might expect this moduli space to inherit some vestige of integrability.

Hitchin proves that, for a compact Riemann surface and certain bundles, a solution is determined just by the holomorphic data of the holomorphic vector bundle and Higgs field. Thus the study of the moduli space can be reduced to a problem in complex geometry and this is the approach that is adopted in this article. Naively the moduli space can be thought of as the cotangent bundle of the moduli space of holomorphic vector bundles on Σ as the Higgs fields are Serre-dual to deformations of the complex structure on a holomorphic vector bundle. Thus the Higgs bundle moduli space is a complex phase space. Furthermore, the coefficients of the characteristic polynomial of the Higgs field can be thought of as defining a system of commuting Hamiltonians and so one has a complex (holomorphic) integrable system which turns out to be algebraic. However, there are a number of technicalities concerning stability and semi-stability that need to be addressed to make these ideas precise, and render the above discussion heuristic.

In keeping with the expository aim of the lectures, the bulk of these notes concern not the theorem and its applications, but the many ingredients which go into its proof. Students with a fairly modest background in geometry should be able to work through these notes, learning a fair amount of algebraic geometry and symplectic geometry along the way, and may be motivated to follow some of the leads in the last section towards open problems and further development of the subject.

1.2.3 The $\bar\partial$ dressing method and integrable geometries, P. Santini

In the previous contributions, it can be seen that a prominent role is played by complex structures. One way of formulating a complex structure is in the form of a $\bar\partial$-operator and, in the case of the Ward transform, the inverse transform from twistor data to the solution on space-time requires the solution of a linear $\bar\partial$-equation. Dressing can be understood as a process by which one takes the transform for a well understood, perhaps trivial, solution where all the ingredients of the tansform are known, and then change the $\bar\partial$-data that appears in the $\bar\partial$-equation to give a more general solution (perhaps the general solution). Over the last few decades such methods have been developed (independently of twistor theory) and extended to include a non-local element in the $\bar\partial$-equation, so that the source term in the $\bar\partial$-equation is given by integrating against a kernel. These non-local terms seem to be essential for certain systems in 2+1 dimensions such as the KP equations etc..

In this contribution the $\bar\partial$-dressing method is shown to apply to certain integrable geometric structures: quadrilateral lattices, a discrete system consisting of lattices in which each elementary quadrilateral is planar, and its continuous limit, the conjugate net, a system studied by Darboux.

The connection between the $\bar{\partial}$-dressing method and these integrable geometries relies upon the following facts:

(1) the simple, linear dependence of the $\bar{\partial}$ data on the coordinates, described by the given linear differential and/or difference equations, defines some basic elementary singularities in the complex plane of the spectral parameter λ (the complex parameter with respect to which the $\bar{\partial}$-problem is defined): essential singularities, poles and branch points, in which the coordinates appear as parameters of the essential singularities, positions of the poles and strength of the branch points.

(2) These elementary singularities and their defining equations have often an elementary and basic geometric meaning. For instance, (a) the matrix equation $\psi_{0x} = i\lambda\sigma_3\psi_0$ and its solution $\psi_0(x,\lambda) = \exp(i\lambda x\sigma_3)$ define the Frenet frame of a straight line in $\mathcal{R}^3$, parallel to the third axis with constant torsion λ and arclength x; (b) the vector difference equations: $\Delta_i\psi_{0j} = 0,\ i = 1,..,N,\ j = 1,..,M$ define the tangent vectors $\psi_{0j} = (0,..,\lambda^{\theta_j},..,0)^T$ of an N - dimensional regular lattice in $\mathcal{R}^M$.

(3) Through the $\bar{\partial}$ dressing method the above basic elementary functions ψ_0 get dressed into new functions ψ which satisfy dressed linear equations in configuration space, whose integrability conditions are the integrable nonlinear systems. In this dressing procedure, the original geometric meaning is usually preserved and suitably deformed. For instance, the linear equation of example (a) is dressed up into $\psi_x = (i\lambda\sigma_3 + Q)\psi$ and describes an arbitrary curve in $\mathcal{R}^3$; while the linear equations of example (b) are dressed up into the linear equations $\Delta_i\psi_j = q_{ji}\psi_i,\ i = 1,..,N,\ j = 1,..,M$ which describe the *planarity* of the elementary quadrilaterals of the N-dimensional lattice (what we call: *a quadrilateral, or planar lattice*).

(4) The associated $\bar{\partial}$ problem provides at the same time:

(i) large classes of solutions of the above geometries, which can therefore be called “integrable”;

(ii) geometrically distinguished symmetry transformations and symmetry reductions of the above geometries.

1.2.4 Differential equations featuring many periodic solutions, F. Calogero

The contribution of Francesco Calogero, who is the originator of modern super-integrable systems amongst many other things, shows a way to obtain evolutionary PDEs which possess many periodic solutions. This development has obvious potential in the context of applications (especially in the modelling of periodic phenomena), but it also sheds light (as more fully shown in other papers by Calogero and others) on a rather fundamental question: the connection between the integrability of evolution equations and the analyticity in complex time of the solutions of such equations, an issue related to the 'Painlevé property'.

1.3 Conclusion

There are many areas of interaction between geometry and integrability that have not been touched on here — the infinite-dimensional grassmanians of Segal & Wilson, the theory of quaternion-Kahler manifolds, the various special integrable classes of two-surfaces embedded into symmetric spaces and so on, but it is to be hoped that these articles will stimulate the reader into further study.

Acknowledgements: I am grateful to Professor Nutku and TUBITAK for the invitation to the Feza Gürsey institute and for their generous hospitality. I should also like to thank Professor Nutku and the contributors for helpful paragraphs in writing this introduction.