

## GEOMETRY AND INTEGRABILITY

Many integrable systems owe their origin to problems in geometry and all are perhaps best understood in a geometrical context. This is especially true today when the heroic early days of study of KdV-type integrability are over. The problems that can be solved using the inverse scattering transformation are now well studied and there are diminishing returns in this direction. Two major techniques have emerged more recently for dealing with multi-dimensional integrable systems: Twistor theory and the  $d$ -bar method, both of which form the subject of this book. It is intended to be an introduction, though by no means an elementary one, to current research on integrable systems in the framework of differential geometry and algebraic geometry.

This book arose from a semester, held at the Feza Gursey Institute, to introduce advanced graduate students to this area of research. The articles are all written by leading researchers and are designed to introduce the reader to contemporary research topics.

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Edited by

Lionel Mason  
*University of Oxford*

and

Yavuz Nutku  
*Feza Gurzey Institute, Istanbul*



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## Contributors

- F. Calogero  
*Università di Roma 'La Sapienza'*
- R. Y. Donagi  
*University of Pennsylvania*
- L. J. Mason  
*The Mathematical Institute, Oxford*
- P. M. Santini  
*Università di Roma 'La Sapienza'*
- K. P. Tod  
*The Mathematical Institute, Oxford*
- N. M. J. Woodhouse  
*The Mathematical Institute, Oxford*



## Preface

Integrable systems continue to fascinate because they are examples of systems with nontrivial nonlinearities that one can nevertheless systematically analyse and often solve exactly analytically. However, there is no royal road to complete integrability, or even a precise all-encompassing definition and so, instead, one must resort to patterns and themes. This volume is concerned with a theme that emerges time and again of the deep links that integrability has with geometry. The motivation for holding a research semester devoted to ‘Geometry and Integrability’ at the Feza Gürsey Institute was precisely for the purpose of exposing students and post-docs to modern geometrical structures that form the natural setting for completely integrable systems.