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Lectures on Invariant Theory

Igor Dolgachev
University of Michigan

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To Natasha

Preface

This book is based on one-semester graduate courses I gave at Michigan in 1994 and 1998, and at Harvard in 1999. A part of the book is borrowed from an earlier version of my lecture notes which were published by the Seoul National University [22]. The main changes consist of including several chapters on algebraic invariant theory, simplifying and correcting proofs, and adding more examples from classical algebraic geometry. The last Lecture of [22], which contains some applications to construction of moduli spaces, has been omitted. The book is literally intended to be a first course in the subject to motivate a beginner to study more. A new edition of D. Mumford's book *Geometric Invariant Theory* with appendices by J. Fogarty and F. Kirwan [74] as well as a survey article of V. Popov and E. Vinberg [90] will help the reader to navigate in this broad and old subject of mathematics. Most of the results and their proofs discussed in the present book can be found in the literature. We include some of the extensive bibliography of the subject (with no claim for completeness). The main purpose of this book is to give a short and self-contained exposition of the main ideas of the theory. The sole novelty is including many examples illustrating the dependence of the quotient on a linearization of the action as well as including some basic constructions in toric geometry as examples of torus actions on affine space. We also give many examples related to classical algebraic geometry. Each chapter ends with a set of exercises and bibliographical notes. We assume only minimal prerequisites for students: a basic knowledge of algebraic geometry covered in the first two chapters of Shafarevich's book [103] and/or Hartshorne's book [46], a good knowledge of multilinear algebra and some rudiments of the theory of linear representations of groups. Although we often use some of the theory of affine algebraic groups, the knowledge of the group GL_n is enough for our purpose.

I am grateful to some of my students and colleagues for critical remarks and catching numerous mistakes in my lecture notes. Special thanks go to Ana-Maria Castravet, Mihnea Popa, Janis Stipins and Ivan Arzhantsev.

Contents

Introduction

Geometric invariant theory arises in an attempt to construct a quotient of an algebraic variety X by an algebraic action of a linear algebraic group G. In many applications X is the parametrizing space of certain geometric objects (algebraic curves, vector bundles, etc.) and the equivalence relation on the objects is defined by a group action. The main problem here is that the quotient space X/G may not exist in the category of algebraic varieties. The reason is rather simple. Since one expects that the canonical projection $f : X \to X/G$ is a regular map of algebraic varieties and so has closed fibres, all orbits must be closed subsets in the Zariski topology of X. This rarely happens when G is not a finite group. A possible solution to this problem is to restrict the action to an invariant open Zariski subset U, as large as possible, so that $U \to U/G$ exists. The geometric invariant theory (GIT) suggests a method for choosing such a set so that the quotient is a quasi-projective algebraic variety. The idea goes back to David Hilbert. Suppose $X = V$ is a linear space and G is a linear algebraic group acting on V via its linear representation. The set of polynomial functions on V invariant with respect to this action is a commutative algebra A over the ground field. Hilbert proves that A is finitely generated if $G = \mathrm{SL}_n$ or GL_n and any set of generators $f_1, \ldots, f_N$ of A defines an invariant regular map from X to some affine algebraic variety Y contained in affine space $\mathbb{A}^N$ whose ring of polynomial functions is isomorphic to A. By a theorem of Nagata the same is true for any reductive linear algebraic group. The map $f : X \to Y$ has a universal property for G-invariant maps of X and is called the categorical quotient. The inverse image of the origin is the closed subvariety defined by all invariant homogeneous polynomials of positive degree. It is called the null-cone. Its points cannot be distinguished by invariant functions; they are called unstable points. The remaining points are called semi-stable points. When we pass to the projective space $\mathbb{P}(V)$ associated to V, the images of semi-stable points form an invariant open subset $\mathbb{P}(V)^{\mathrm{ss}}$ and the map f induces a regular map $\bar{f} : \mathbb{P}(V)^{\mathrm{ss}} \to \bar{Y}$, where $\bar{Y}$ (denoted by $\mathbb{P}(V)^{\mathrm{ss}}/\!\!/G$) is a projective algebraic variety with the projective coordinate algebra isomorphic to A. In applications considered by Hilbert, $\mathbb{P}(V)$ parametrizes projective hypersurfaces of certain degree and dimension, and the projective algebraic variety $\bar{Y}$ is the "moduli space" of these hypersurfaces. The hypersurfaces represented by unstable points are left out from the moduli space; they are "too degenerate". A nonsingular hypersurface is always represented by a semi-stable point. Since $\bar{Y}$ is a projective variety, it is considered as a "compactification" of the moduli space of nonsingular hypersurfaces. The fibres of the map $\mathbb{P}(V)^{\rm ss} \to \mathbb{P}(V)^{\rm ss}/\!\!/G$ are not orbits in general; however, each fibre contains a unique closed orbit so that $\mathbb{P}(V)^{\rm ss}/\!\!/G$ parametrizes closed orbits in the set of semi-stable points.

Since the equations of the null-cone are hard to find without computing explicitly the ring of invariant polynomials, one uses another approach. This approach is to describe the set of semi-stable points by using the Hilbert–Mumford numerical criterion of stability. In many cases it allows one to determine the set $\mathbb{P}(V)^{\rm ss}$ very explicitly and to distinguish stable points among semi-stable ones. These are the points whose orbits are closed in $\mathbb{P}(V)^{\rm ss}$ and whose stabilizer subgroups are finite. The restriction of the map $\mathbb{P}(V)^{\rm ss} \to \mathbb{P}(V)^{\rm ss}/\!\!/G$ to the set of stable points $\mathbb{P}(V)^{\rm s}$ is an orbit map $\mathbb{P}(V)^{\rm s} \to \mathbb{P}(V)^{\rm s}/G$. It is called a geometric quotient.

More generally, if G is a reductive algebraic group acting on a projective algebraic variety X, the GIT approach to constructing the quotient consists of the following steps. First one chooses a linearization of the action, a G-equivariant embedding of X into a projective space $\mathbb{P}(V)$ with a linear action of G as above. The choice of a linearization is a parameter of the construction; it is defined by a G-linearized ample line bundle on X. Then one sets $X^{\rm ss} = X \cap \mathbb{P}(V)^{\rm ss}$ and defines the categorical quotient $X^{\rm ss} \to X^{\rm ss}/\!\!/G$ as the restriction of the categorical quotient $\mathbb{P}(V)^{\rm ss} \to \mathbb{P}(V)^{\rm ss}/\!\!/G$. The image variety $X^{\rm ss}/\!\!/G$ is a closed subvariety of $\mathbb{P}(V)^{\rm ss}/\!\!/G$.

Let us give a brief comment on the content of the book.

In Chapters 1 and 2 we consider the classical example of invariant theory in which the general linear group $\mathrm{GL}(V)$ of a vector space V of dimension n over a field k acts naturally on the space of homogeneneous polynomials $\mathrm{Pol}_d(V)$ of some degree d. We explain the classical symbolic method which allows one to identify an invariant polynomial function of degree m on this space with an element of the projective coordinate algebra $k[\mathrm{Gr}(n,m)]$ on the Grassmann variety $\mathrm{Gr}(n,m)$ of n-dimensional linear subspaces in k^m in its Plücker embedding. This interpretation is based on the First Fundamental Theorem of Invariant Theory. The proof of this theorem uses a rather technical algebraic tool, the so-called Clebsch omega-operator. We choose this less conceptual approach to show the flavor of the invariant theory of the nineteenth century. More detailed expositions of the classical invariant theory ([64], [122]) give a conceptual explanation of this operator via representation theory. The Second Fundamental Theorem of Invariant Theory is just a statement about the relations between the Plücker coordinates known in algebraic geometry as the Plücker equations. We use the available computations of invariants in later chapters to give an explicit description of some of the GIT quotients arising in classical algebraic geometry.

In Chapter 3 we discuss the problem of finite generatedness of the algebra of invariant polynomials on the space of a linear rational representation of an algebraic group. We begin with the Gordan–Hilbert theorem and explain the "unitary trick" due to Adolf Hurwitz and Hermann Weyl which allows one to prove the finite generatedness in the case of a semisimple or, more generally, reductive complex algebraic group. Then we introduce the notion of a geometrically reductive algebraic group and prove Nagata's theorem on finite generatedness of the algebra of invariant polynomials on the space of a linear rational representation of a reductive algebraic group.

In Chapter 4 we discuss the case of a linear rational representation of a nonreductive algebraic group. We prove a lemma due to Grosshans which allows one to prove finite generatedness for the restriction of a representation of a reductive algebraic group G to a subgroup H provided the algebra of regular functions on the homogeneous space G/H is finitely generated. A corollary of this result is a classical theorem of Weitzenböck about invariants of the additive group. The central part of this chapter is Nagata's counterexample to Hilbert's Fourteenth Problem. It asks about finite generatedness of the algebra of invariants for an arbitrary algebraic group of linear transformations. We follow the original construction of Nagata with some simplifications due to R. Steinberg.

Chapter 5 is devoted to covariants of an action. A covariant of an affine algebraic group G acting on an algebraic variety X is a G-equivariant regular map from X to an affine space on which the group acts via its linear representation. The covariants form an algebra and the main result of the theory is that this algebra is finitely generated if G is reductive. The proof depends heavily on the theory of linear representations of reductive algebraic groups which we review in this chapter. As an application of this theory we prove the classical Cayley-Sylvester formula for the dimension of the spaces of covariants and also the Hermite reciprocity.

In Chapter 6 we discuss categorical and geometric quotients of an algebraic variety under a regular action of an algebraic group. The material is fairly standard and follows Mumford's book.

Chapter 7 is devoted to linearizations of actions. The main result is that any algebraic action of a linear algebraic group on a normal quasi-projective algebraic variety X is isomorphic to the restriction of a linear action on a projective space in which X is equivariantly embedded. The proof follows the exposition of the theory of linearizations from [65].

Chapter 8 is devoted to the concept of stability of algebraic actions and the construction of categorical and geometric quotients. The material of this chapter is rather standard and can be found in Mumford's book as well as in many other books. We include many examples illustrating the dependence of the quotients on the linearization.

Chapter 9 contains the proof of Hilbert and Mumford's numerical criterion of stability. The only novelty here is that we also include Kempf's notion of stability and give an example of its application to the theory of moduli of abelian varieties.

The remaining chapters 10–12 are devoted to some examples where the complete description of stable points is available. In Chapter 10 we discuss the case of hypersurfaces in projective space. We give explicit descriptions of the moduli spaces of binary forms of degree ≤ 5, plane curves of degree 3 and cubic surfaces. In Chapter 11 we discuss moduli spaces of ordered collections of linear subspaces in projective space, in particular of points in $\mathbb{P}^n$ or of lines in $\mathbb{P}^3$. The examples discussed in this chapter are related to some of the beautiful constructions of classical algebraic geometry. In Chapter 12 we introduce toric varieties as GIT quotients of an open subset of affine space. Some of the constructions discussed in the preceding chapters admit a nice interpretation in terms of the geometry of toric varieties. This approach to toric varieties is based on some recent work of D. Cox ([16]) and M. Audin ([3]).

We will be working over an algebraically closed field k sometimes assumed to be of characteristic zero.