Chapter 1

The symbolic method

1.1 First examples

The notion of an invariant is one of the most general concepts of mathematics. Whenever a group G acts on a set S we look for elements $s \in S$ which do not change under the action, i.e., which satisfy $g \cdot s = s$ for any $g \in G$. For example, if S is a set of functions from a set X to a set Y, and G acts on S via its action on X and its action on Y by the formula

$$(g \cdot f)(x) = g \cdot f(g^{-1} \cdot x),$$

then an *equivariant function* is a function $f : X \to Y$ satisfying $g \cdot f = f$, i.e.,

$$f(g \cdot x) = g \cdot f(x), \quad \forall g \in G, \forall x \in X.$$

In the case when G acts trivially on Y, an equivariant function is called an *invariant function*. It satisfies

$$f(g \cdot x) = f(x), \quad \forall g \in G, \forall x \in X.$$

Among all invariant functions there exists a universal function, the projection map $p : X \to X/G$ from the set X to the set of orbits X/G. It satisfies the property that for any invariant function $f : X \to Y$ there exists a unique map $\bar{f} : X/G \to Y$ such that $f = \bar{f} \circ p$. So if we know the set of orbits X/G, we know all invariant functions on X. We will be concerned with invariants arising in algebra and algebraic geometry. Our sets and our group G will be algebraic varieties and our invariant functions will be regular maps.

Let us start with some examples.

Example 1.1. Let A be a finitely generated algebra over a field k and let G be a group of its automorphisms. The subset

$$A^G = \{a \in A : g(a) = a, \forall g \in G\} \tag{1.1}$$

is a k-subalgebra of A. It is called the *algebra of invariants*. This definition fits the general setting if we let $X = \mathrm{Spm}(A)$ be the affine algebraic variety over k with coordinate ring equal to A, and let $Y = \mathbb{A}^1_k$ be the affine line over k. Then elements of A can be viewed as regular functions $a : X \to \mathbb{A}^1_k$ between algebraic varieties. A more general invariant function is an invariant map $f : X \to Y$ between algebraic varieties. If Y is affine with coordinate ring B, such a map is defined by a homomorphism of k-algebras $f^* : B \to A$ satisfying $g(f^*(b)) = f^*(b)$ for any $g \in G, b \in B$. It is clear that such a homomorphism is equal to the composition of a homomorphism $B \to A^G$ and the natural inclusion map $A^G \to A$. Thus if we take $Z = \mathrm{Spm}(A^G)$ we obtain that the map $X \to Z$ defined by the inclusion $A^G \hookrightarrow A$ plays the role of the universal function. So it is natural to assume that A^G is the coordinate ring of the orbit space X/G. However, we shall quickly convince ourselves that there must be some problems here. The first one is that the algebra A^G may not be finitely generated over k and so does not define an algebraic variety. This problem can be easily resolved by extending the category of algebraic varieties to the category of schemes. For any (not necessarily finitely generated) algebra A over k, we may still consider the *subring of invariants* A^G and view any homomorphism of rings $B \to A$ as a morphism of affine schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$. Then the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A^G)$ is the universal invariant function. However, it is preferable to deal with algebraic varieties rather than to deal with arbitrary schemes, and we will later show that A^G is always finitely generated if the group G is a reductive algebraic group which acts algebraically on $\mathrm{Spm}(A)$. The second problem is more serious. The affine algebraic variety $\mathrm{Spm}(A^G)$ rarely coincides with the set of orbits (unless G is a finite group). For example, the standard action of the general linear group $\mathrm{GL}_n(k)$ on the space k^n has two orbits but no invariant nonconstant functions.

The following is a more interesting example.

Example 1.2. Let $G = \mathrm{GL}_n(k)$ act by automorphisms on the polynomial algebra $A = k[X_{11}, \ldots, X_{nn}]$ in n^2 variables $X_{ij}, i, j = 1, \ldots, n$, as follows. For any $g = (a_{ij}) \in G$ the polynomial $g(X_{ij})$ is equal to the ijth entry of the matrix

$$Y = g^{-1} \cdot X \cdot g, \tag{1.2}$$

where $X = (X_{ij})$ is the matrix with the entries X_{ij}. Then, the affine variety $\mathrm{Spm}(A)$ is the affine space Mat_n of dimension n^2. Its k-points can be interpreted as $n \times n$ matrices with entries in k and we can view elements of A as polynomial functions on the space of matrices. We know from linear algebra that any such matrix can be reduced to its Jordan form by means of a transformation (1.2) for an appropriate g. Thus any invariant function is uniquely determined by its values on Jordan matrices. Let D be the subspace of diagonal matrices identified with linear space k^n and let $k[\Lambda_1, \ldots, \Lambda_n]$ be the algebra of polynomial functions on D. Since the set of matrices with diagonal Jordan form is a Zariski dense subset in the set of all matrices, we see that an invariant function is uniquely determined by its values on diagonal matrices. Therefore the restriction homomorphism $A^G \to k[\Lambda_1, \ldots, \Lambda_n]$ is injective. Since two diagonal matrices with permuted diagonal entries are equivalent, an invariant function must be a symmetric polynomial in Λ_i. By the Fundamental Theorem on Symmetric Functions, such a function can be written uniquely as a polynomial in elementary symmetric functions s_i in the variables $\Lambda_1, \ldots, \Lambda_n$. On the other hand, let c_i be the coefficients of the characteristic polynomial

$$\det(X - tI_n) = (-1)^n t^n + c_1(-t)^{n-1} + \cdots + c_n$$

considered as polynomial functions on Mat_n, i.e., elements of the ring A. Clearly, the restriction of c_i to D is equal to the ith elementary symmetric function s_i. So we see that the image of A^G in $k[\Lambda_1, \ldots, \Lambda_n]$ coincides with the polynomial subalgebra $k[s_1, \ldots, s_n]$. This implies that A^G is freely generated by the functions c_i. So we can identify $\mathrm{Spm}(A^G)$ with affine space k^n. Now consider the universal map $\mathrm{Spm}(A) \to \mathrm{Spm}(A^G)$. Its fibre over the point $(0, \ldots, 0)$ defined by the maximal ideal $(c_1, \ldots, c_n)$ is equal to the set of matrices M with characteristic polynomial $\det(M - tI_n) = (-t)^n$. Clearly, this set does not consist of one orbit, any Jordan matrix with zero diagonal values belongs to this set. Thus $\mathrm{Spm}(A^G)$ is not the orbit set $\mathrm{Spm}(A)/G$.

We shall discuss later how to remedy the problem of the construction of the space of orbits in the category of algebraic varieties. This is the subject of the geometric invariant theory (GIT) with which we will be dealing later. Now we shall discuss some examples where the algebra of invariants can be found explicitly.

Let E be a finite-dimensional vector space over a field k and let

$$\rho : G \to \mathrm{GL}(E)$$

be a linear representation of a group G in E. We consider the associated action of G on the space $\mathrm{Pol}_m(E)$ of degree m homogeneous polynomial functions on E. This action is obviously linear. The value of $f \in \mathrm{Pol}_m(E)$ at a vector v is given, in terms of the coordinates $(t_1, \ldots, t_r)$ of v with respect to some basis $(\xi_1, \ldots, \xi_r)$, by the following expression:

$$f(t_1, \ldots, t_r) = \sum_{\substack{i_1,\ldots,i_r \geq 0 \\ i_1+\cdots+i_r=m}} a_{i_1\ldots i_r} t_1^{i_1} \cdots t_r^{i_r},$$

or in the vector notation,

$$f(\mathbf{t}) = \sum_{\substack{\mathbf{i}\in\mathbb{Z}^r_{\geq 0} \\ |\mathbf{i}|=m}} a_{\mathbf{i}} \mathbf{t}^{\mathbf{i}}. \tag{1.3}$$

The direct sum of the vector spaces $\mathrm{Pol}_m(E)$ is equal to the graded algebra of polynomial functions $\mathrm{Pol}(E)$. Since k is infinite (we assumed it to be algebraically closed), $\mathrm{Pol}(E)$ is isomorphic to the polynomial algebra $k[T_1, \ldots, T_r]$. In more sophisticated language, $\mathrm{Pol}_m(E)$ is naturally isomorphic to the mth symmetric product $S^m(E^*)$ of the dual vector space E^* and $\mathrm{Pol}(E)$ is isomorphic to the symmetric algebra $S(E^*)$.

We will consider the case when $E = \mathrm{Pol}_d(V)$ and $G = \mathrm{SL}(V)$ be the special linear group with its linear action on E described above. Let $A = \mathrm{Pol}(\mathrm{Pol}_d(V))$. We can take for coordinates on the space $\mathrm{Pol}_d(V)$ the functions $A_{\mathbf{i}}$ which assign to a homogeneous form (1.3) its coefficient $a_{\mathbf{i}}$. So any element from A is a polynomial in the $A_{\mathbf{i}}$. We want to describe the subalgebra of invariants A^G.

The problem of finding A^G is almost two centuries old. Many famous mathematicians of the nineteenth century made a contribution to this problem. Complete results, however, were obtained only in a few cases. The most complete results are known in the case $\dim V = 2$, the case where E consists of *binary forms* of degree d. We write a binary form as

$$p(t_0, t_1) = a_0 t_0^d + a_1 t_0^{d-1} t_1 + \cdots + a_d t_1^d.$$

In this case we have $d + 1$ coefficients, and hence elements of A are polynomials $P(A_0, \ldots, A_d)$ in $d + 1$ variables.

1.2 Polarization and restitution

To describe the ring $\mathrm{Pol}(\mathrm{Pol}_d(V))^{\mathrm{SL}(V)}$ one uses the symbolic expression of a polynomial, which we now explain. We assume that $\mathrm{char}(k) = 0$.

A homogeneous polynomial of degree 2 on a vector space E is a quadratic form. Recall its coordinate-free definition: a map $Q : E \to k$ is a quadratic form if the following two properties are satisfied:

(i) $Q(tv) = t^2 Q(v)$, for any $v \in E$ and any $t \in k$;

(ii) the map $\tilde{Q} : E \times E \to k$ defined by the formula

$$\tilde{Q}(v, w) = Q(v + w) - Q(v) - Q(w)$$

is bilinear.

A homogeneous polynomial $P \in \mathrm{Pol}_m(E)$ of degree m can be defined in a similar way by the following properties:

(i) $P(tv) = t^m P(v)$, for any $v \in E$ and any $t \in k$;

(ii) the map $\mathrm{pol}(P) : E^m \to k$ defined by the formula

$$\mathrm{pol}(P)(v_1, \ldots, v_m) = \sum_{I \subset [m]} (-1)^{m - \# I} P\left(\sum_{i \in I} v_i\right)$$

is multilinear.

Here and throughout we use $[m]$ to denote the set $\{1, \ldots, m\}$.

As in the case of quadratic forms, we immediately see that the map $\mathrm{pol}(P)$ is a symmetric multilinear form and also that P can be reconstructed from $\mathrm{pol}(P)$ by the formula

$$m! P(v) = \mathrm{pol}(P)(v, \ldots, v).$$

The symmetric multilinear form $\mathrm{pol}(P)$ is called the *polarization* of P. For any symmetric multilinear from $F : E^m \to k$ the function $\mathrm{res}(F) : E \to k$ defined by

$$\mathrm{res}(F)(v) = F(v, \ldots, v)$$

is called the *restitution* of F. It is immediately checked that $\mathrm{res}(F) \in \mathrm{Pol}_m(V)$ and

$$\mathrm{pol}(\mathrm{res}(F)) = m! F.$$

Since we assumed that $\mathrm{char}(k) = 0$, we obtain that each $P \in \mathrm{Pol}_m(E)$ is equal to the restitution of a unique symmetric m-multilinear form, namely $\frac{1}{m!}\mathrm{pol}(P)$.

Assume that P is equal to the product of linear forms $P = L_1 \ldots L_m$. We have

$$\mathrm{pol}(P)(v_1, \ldots, v_m) = \sum_{I \subset [m]} (-1)^{m - \# I} L_1 \ldots L_m \Big(\sum_{i \in I} v_i\Big)$$

$$
\begin{aligned}
&= \sum_{I\subset[m]} (-1)^{m-\#I} L_1\Big(\sum_{i\in I} v_i\Big)\dots L_m\Big(\sum_{i\in I} v_i\Big) \\
&= \sum_{I\subset[m]} (-1)^{m-\#I} \Big(\sum_{i\in I} L_1(v_i)\Big)\dots\Big(\sum_{i\in I} L_m(v_i)\Big) \\
&= \sum_{\sigma\in\Sigma_m} L_1(v_{\sigma(1)}\dots L_m(v_{\sigma(m)}) = \sum_{\sigma\in\Sigma_m} L_{\sigma(1)}(v_1)\dots L_{\sigma(m)}(v_m) \qquad (1.4)
\end{aligned}
$$

Here Σ_m denotes the permutation group on m letters.

Let $(\xi_1,\dots,\xi_n)$ be a basis of E and $(t_1,\dots,t_n)$ be the dual basis of E^*. Any $v\in E$ can be written in a unique way as $v=\sum_{i=1}^n t_i(v)\xi_i$. Let $\mathrm{Sym}_m(E)$ be the vector space of symmetric m-multilinear forms on E^m. For any $v_1,\dots,v_m\in E$ and any $F\in\mathrm{Sym}_m(E)$, we have

$$
\begin{aligned}
F(v_1,\dots,v_m) &= F\left(\sum_{i=1}^n t_i(v_1)\xi_i,\dots,\sum_{i=1}^n t_i(v_m)\xi_i\right) \\
&= \sum_{i_1,\dots,i_m=1}^n t_{i_1}(v_1)\dots t_{i_m}(v_m)F(\xi_{i_1},\dots,\xi_{i_m}).
\end{aligned}
$$

Taking $v_1=\dots=v_m=v$, we obtain that

$$
\begin{aligned}
\mathrm{res}(F)(v) &= \sum_{i_1,\dots,i_m=1}^n t_{i_1}(v)\dots t_{i_m}(v)F(\xi_{i_1},\dots,\xi_{i_m}) \\
&= \left(\sum_{i_1,\dots,i_m=1}^n a_{i_1\dots i_m} t_{i_1}\dots t_{i_m}\right)(v).
\end{aligned}
$$

Thus any polynomial $P\in\mathrm{Pol}_m(E)$ can be written uniquely as a sum of monomials $t_{i_1}\dots t_{i_m}$. This is the coordinate-dependent definition of a homogeneous polynomial. Since the *polarization map*

$$\mathrm{pol}:\mathrm{Pol}_m(E)\to\mathrm{Sym}_m(E)$$

is obviously linear, we obtain that $\mathrm{Sym}_m(E)$ has a basis formed by the polarizations of monomials $t_{i_1}\dots t_{i_m}$. Applying (1.4), we have

$$\mathrm{pol}(t_{i_1}\dots t_{i_m})(v_1,\dots,v_m) = \sum_{\sigma\in\Sigma_m} t_{\sigma(1)}(v_1)\dots t_{\sigma(m)}(v_m).$$

If we denote by $(t_1^{(j)},\ldots,t_n^{(j)})$ a jth copy of the basis $(t_1,\ldots,t_n)$ in E^*, we can rewrite the previous expression as

$$\mathrm{pol}(t_{i_1}\ldots t_{i_m})(v_1,\ldots,v_m) = \sum_{\sigma\in\Sigma_m} t^{(1)}_{\sigma(1)}\ldots t^{(m)}_{\sigma(m)}(v_1,\ldots,v_m).$$

Here, we consider the product of m linear forms on V as an m-multilinear form on E^m. We have

$$\mathrm{pol}(t_{i_1}\ldots t_{i_m})(\xi_{j_1},\ldots,\xi_{j_m}) = \#\{\sigma\in\Sigma_m : (j_1,\ldots,j_m) = (i_{\sigma(1)},\ldots,i_{\sigma(m)})\}. \tag{1.5}$$

If we write $t_{i_1}\ldots t_{i_m} = t_1^{k_1}\ldots t_n^{k_n}$, then the right-hand side is equal to $k_1!\ldots k_n!$ if $\{i_1,\ldots,i_m\} = \{j_1,\ldots,j_m\}$ and zero otherwise.

Note that the polarization allows us to identify $\mathrm{Pol}_m(E)$ with the dual to the space $\mathrm{Pol}_m(E^*)$. To see this, choose a basis of $\mathrm{Pol}_m(E^*)$ formed by the monomials $\xi_{i_1}\ldots\xi_{i_m}$. For any $F\in\mathrm{Sym}_m(E)$ we can set

$$F(\xi_{i_1}\ldots\xi_{i_m}) = F(\xi_{i_1},\ldots,\xi_{i_m})$$

and then extend the domain of F to all homogeneous degree m polynomials by linearity. Applying (1.5), we get

$$\mathrm{pol}(t_1^{k_1}\ldots t_n^{k_m})(\xi_1^{l_1}\ldots\xi_n^{l_n}) = \begin{cases} k_1!\ldots k_n! & \text{if } (k_1,\ldots,k_n) = (l_1,\ldots,l_n),\\ 0 & \text{otherwise.}\end{cases}$$

This shows that the map from $\mathrm{Pol}_m(E)\times\mathrm{Pol}_m(E^*)$ to k defined by

$$(P,Q) = \frac{1}{m!}\mathrm{pol}(P)(Q) \tag{1.6}$$

is a perfect duality, i.e., it defines isomorphisms

$$\mathrm{Pol}_m(E)^*\cong\mathrm{Pol}_m(E^*),\quad \mathrm{Pol}_m(E^*)^*\cong\mathrm{Pol}_m(E). \tag{1.7}$$

Moreover, the monomial basis $(\boldsymbol{\xi}^{\mathbf{k}}) = (\xi_1^{k_1}\ldots\xi_n^{k_n})$ of $\mathrm{Pol}_m(E^*)$ is dual to the basis $(\frac{m!}{k_1!\ldots k_n!}t_1^{k_1}\ldots t_n^{k_n}) = (\frac{m!}{\mathbf{k}!}\mathbf{t}^{\mathbf{k}})$.

Remark 1.1. Note that the coefficients $a_{\mathbf{k}}$ of a polynomial

$$P = \sum_{|\mathbf{k}|=m}\frac{m!}{\mathbf{k}!}a_{\mathbf{k}}\mathbf{t}^{\mathbf{k}}\in\mathrm{Pol}_m(E) \tag{1.8}$$

are equal to the value of $A_{\mathbf{k}} = \xi^{\mathbf{k}} = \xi_1^{k_1} \ldots \xi_n^{k_n}$ on P. We can view the expression $P_{\text{general}} = \sum_{|\mathbf{k}|=m} \frac{m!}{\mathbf{k}!} A_{\mathbf{k}} \mathbf{t}^{\mathbf{k}}$ as a "general" homogeneous polynomial of degree m. Thus we get a strange formula

$$P_{\text{general}} = \sum_{|\mathbf{k}|=m} \frac{m!}{\mathbf{k}!} A_{\mathbf{k}} \mathbf{t}^{\mathbf{k}} = \sum_{|\mathbf{k}|=m} \frac{m!}{\mathbf{k}!} \xi^{\mathbf{k}} \mathbf{t}^{\mathbf{k}} = \left(\sum_{i=1}^{n} \xi_i t_i \right)^m .$$

This explains the classical notation of a homogeneous polynomial as a power of a linear polynomial.

Remark 1.2. One can view a basis vector ξ_i as a linear differential operator on $\mathrm{Pol}(E)$ which acts on linear functions by $\xi_i(t_j) = \delta_{ij}$. It acts on any polynomial $P = \sum_{\mathbf{k}} a_{\mathbf{k}} \mathbf{t}^{\mathbf{k}}$ as the partial derivative $\partial_i = \frac{\partial}{\partial t_i}$. Thus we can identify any polynomial $D(t_1, \ldots, t_n) \in \mathrm{Pol}(E^*)$ with the differential operator $\tilde{D}(\partial_1, \ldots, \partial_n)$ by replacing the variable ξ_i with ∂_i. In this way the duality $\mathrm{Pol}_m(E^*) \times \mathrm{Pol}_m(E) \to k$ is defined by the formula

$$(D, P) = \frac{1}{m!} \tilde{D}(P).$$

Remark 1.3. For the reader with a deeper knowledge of multilinear algebra, we recall that there is a natural isomorphism between the linear space $\mathrm{Pol}_m(E)$ and the mth symmetric power $S^m(E^*)$ of the dual space E^*. The polarization map is a linear map from $S^m(E^*)$ to $S^m(E)^*$ which is bijective when $(\mathrm{char}(k), m!) = 1$. The universal property of tensor product allows one to identify the spaces $S^m(E)^*$ and $\mathrm{Sym}_m(E)$.

Let us now consider the case when $E = \mathrm{Pol}_d(V)$, where $\dim V = r$.

First recall that a *multihomogeneous function* of *multi-degree* $(d_1, \ldots, d_m)$ on V is a function on V^m which is a homogeneous polynomial function of degree d_i in each variable; when each $d_i = 1$, we get the usual definition of a multilinear function. We denote the linear space of multihomogeneous functions of multi-degree $(d_1, \ldots, d_m)$ by $\mathrm{Pol}_{d_1,\ldots,d_m}(V)$. The symmetric group Σ_m acts naturally on the space $\mathrm{Pol}_{d,\ldots,d}(V)$ by permuting the variables. The subspace of invariant (symmetric) functions will be denoted by $\mathrm{Sym}_{d,\ldots,d}(V)$. In particular,

$$\mathrm{Sym}_{1,\ldots,1}(V) = \mathrm{Sym}_m(V).$$

Lemma 1.1. *We have a natural isomorphism of linear spaces*

$$\mathrm{symb} : \mathrm{Pol}_m(\mathrm{Pol}_d(V)) \to \mathrm{Sym}_{d,\ldots,d}(V^*).$$

Proof. The polarization map defines an isomorphism

$$\mathrm{Pol}_m(\mathrm{Pol}_d(V)) \cong \mathrm{Sym}_m(\mathrm{Pol}_d(V)).$$

Using the polarization again we obtain an isomorphism $\mathrm{Pol}_d(V)^* \cong \mathrm{Pol}_d(V^*)$. Thus any linear function on $\mathrm{Pol}_d(V)$ is a homogeneous polynomial function of degree d on V^*. Thus a multilinear function on $\mathrm{Pol}_d(V)$ can be identified with a multihomogeneous function on V^* of multi-degree $(d,\dots,d)$. □

Let us make the isomorphism from the preceding lemma more explicit by using a basis $(\xi_1,\dots,\xi_r)$ in V and its dual basis $(t_1,\dots,t_r)$ in V^*. Let $A_{\mathbf{k}}, |\mathbf{k}| = d$, be the coordinate functions on $\mathrm{Pol}_d(V)$, where we write each $P \in \mathrm{Pol}_d(V)$ as in (1.8) with m replaced by d, so that $A_{\mathbf{k}}(P) = a_{\mathbf{k}}$. Any $F \in \mathrm{Pol}_m(\mathrm{Pol}_d(V))$ is a polynomial expression in the $A_{\mathbf{k}}$ of degree m. Let $(A_{\mathbf{k}}^{(1)}),\dots,(A_{\mathbf{k}}^{(m)})$ be the coordinate functions in each copy of $\mathrm{Pol}_d(V)$. The polarization $\mathrm{pol}(F)$ is a multilinear expression in the $A_{\mathbf{k}}^j$. Now, if we replace $A_{\mathbf{k}}^{(j)}$ with the monomial $\boldsymbol{\xi}^{(j)\mathbf{k}}$ in a basis $(\xi_1^{(j)},\dots,\xi_r^{(j)})$ of the jth copy of V, we obtain the *symbolic expression* of F

$$\mathrm{symb}(F)(\boldsymbol{\xi}^{(1)},\dots,\boldsymbol{\xi}^{(m)}) \in \mathrm{Pol}_{d,\dots,d}(V^*).$$

Remark 1.4. The mathematicians of the nineteenth century did not like superscripts and preferred to use different letters for vectors in different copies of the same space. Thus they would write a general polynomial $P = \sum_{\mathbf{k}} \frac{m!}{\mathbf{k}!} A_{\mathbf{k}} \mathbf{t}^{\mathbf{k}}$ of degree d as

$$P = \left(\sum_i \alpha_i t_i\right)^d = \left(\sum_i \beta_i t_i\right)^d = \cdots,$$

and the symbolic expression of a function $F(\dots,A_{\mathbf{k}},\dots)$ as an expression in $\alpha_i, \beta_i, \dots$.

Example 1.3. Let $r=2, d=2$. In this case $\mathrm{Pol}_2(V)$ consists of quadratic forms in two variables $P = a_0x_0^2 + 2a_1x_0x_1 + a_2x_1^2$. The discriminant $D = A_{20}A_{02} - A_{11}^2$ is an obvious invariant of $\mathrm{SL}_2(k)$. We have

$$\mathrm{pol}(D) = A_{20}B_{02} + A_{02}B_{20} - 2A_{11}B_{11},$$
$$\mathrm{symb}(D) = \alpha_0^2\beta_1^2 + \alpha_1^2\beta_0^2 - 2\alpha_0\alpha_1\beta_0\beta_1 = (\alpha_0\beta_1 - \alpha_1\beta_0)^2 = (\alpha,\beta)^2,$$

where

$$(\alpha,\beta) = \det\begin{pmatrix}\alpha_0 & \alpha_1\\ \beta_0 & \beta_1\end{pmatrix}.$$

Example 1.4. Let $r = 2, d = 4$. The determinant (called the *Hankel determinant*)

$$\det \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix}$$

in coefficients of a binary quartic

$$f = a_0x_0^4 + 4a_1x_0^3x_1 + 6a_2x_0^2x_1^2 + 4a_3x_0x_1^3 + a_4x_1^2$$

defines a function $C \in \mathrm{Pol}_3(\mathrm{Pol}_4(k^2))$ on the space of binary quartics. It is called the *catalecticant*. We leave as an exercise to verify that its symbolic expression is equal to

$$\mathrm{symb}(C) = (\alpha, \beta)^2(\alpha, \gamma)^2(\beta, \gamma)^2.$$

It is immediate to see that the group $\mathrm{GL}_2(k)$ acts on $k[a_0, \ldots, a_4]$ via its action on α, β, γ by

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}, \ldots. \tag{1.9}$$

This implies that the catalecticant is invariant with respect to the group $\mathrm{SL}_2(k)$.

1.3 Bracket functions

It is convenient to organize the variables $\xi_1^{(1)}, \ldots, \xi_r^{(1)}, \ldots, \xi_1^{(m)}, \ldots, \xi_r^{(m)}$ as a matrix of size $r \times m$:

$$A = \begin{pmatrix} \xi_1^{(1)} & \ldots & \xi_1^{(m)} \\ \vdots & \ddots & \vdots \\ \xi_r^{(1)} & \ldots & \xi_r^{(m)} \end{pmatrix}.$$

First, we identify the space $\mathrm{Pol}_{d,\ldots,d}(V^*)$ with the subspace of the polynomial algebra $k[\xi_1^{(1)}, \ldots, \xi_r^{(1)}; \ldots; \xi_1^{(m)}, \ldots, \xi_r^{(m)}]$ consisting of polynomials which are homogeneous of degree d in each set of variables $\xi_1^{(j)}, \ldots, \xi_r^{(j)}$. Next, we identify the algebra $k[\xi_1^{(1)}, \ldots, \xi_r^{(1)}; \ldots; \xi_1^{(m)}, \ldots, \xi_r^{(m)}]$ with the algebra $\mathrm{Pol}(\mathrm{Mat}_{r,m})$ of polynomial functions on the space of matrices $\mathrm{Mat}_{r,m}$. The value of a variable $\xi_i^{(j)}$ at a matrix A is the ijth entry of the matrix. The group $(k^*)^m$ acts naturally on the space $\mathrm{Mat}_{r,m}$ by

$$(\lambda_1, \ldots, \lambda_m) \cdot [C_1, \ldots, C_m] = [\lambda_1 C_1, \ldots, \lambda_m C_m],$$