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HYDRODYNAMIC STABILITY

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SECOND EDITION

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TO OUR PARENTS

Isaac and Leah Drazin

William and Edna Reid

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FOREWORD

The study of hydrodynamic stability goes back to the theoretical work of Helmholtz (1868), Kelvin (1871) and Rayleigh (1879, 1880) on inviscid flows and, above all, the experimental investigations of Reynolds (1883), which initiated the systematic study of viscous shear flows. Reynolds's work stimulated the theoretical investigations of Orr (1907) and Sommerfeld (1908), who independently considered small, traveling-wave disturbances of an otherwise steady, parallel flow and derived (what is now known as) the Orr–Sommerfeld equation.

Early attempts to solve the Orr–Sommerfeld equation for the flow associated with the uniform, relative motion of two parallel plates (plane Couette flow) led to the prediction of stability for all Reynolds numbers, in apparent disagreement with experiment (although the prediction *is* correct for infinitesimal disturbances). G. I. Taylor (1923), referring to this work, remarked that:

> This problem has been chosen because it seemed probable that the mathematical analysis might prove comparatively simple: but … it has actually proved very complicated and difficult. [Moreover] it would be extremely difficult to verify experimentally any conclusions which might be arrived at in this case, because of the difficulty of designing apparatus in which the required boundary conditions are approximately satisfied.
>
> It is very much easier to design apparatus for studying the flow of fluid under pressure through a tube, or the flow between two concentric rotating cylinders. The experiments of Reynolds and others suggest that [for] flow through a circular tube, infinitely small disturbances are stable, while larger disturbances increase provided the speed of flow is greater than a certain amount. The study of the fluid stability when the disturbances are not … infinitely small is extremely difficult. It seems more promising therefore to examine the stability of liquid contained between concentric rotating cylinders [circular Couette flow].

Taylor then proceeded to carry out both stability calculations and experiments for circular Couette flow and concluded with the characteristic statement that 'The accuracy with which the observed and calculated sets of points [agree] is remarkable when it is remembered how complicated was the analysis employed in obtaining them.'

As Taylor anticipated, corresponding results for parallel shear flows, for which the Reynolds number at transition is relatively large, proved to be more elusive. An essential difficulty in understanding the instability of such flows is the dual role of viscosity, which may be either stabilizing, as intuitively expected, or destabilizing, a phenomenon adumbrated by Taylor (1915) and explicated by Prandtl (1921) and Tollmien (1929). Moreover, the asymptotic (Reynolds number $\to \infty$) methods invoked by Heisenberg (1924), Tollmien (1929) and Schlichting (1933) to solve the Orr–Sommerfeld equation were heuristic and left doubts about the theoretical predictions, especially as these predictions conflicted with observation. The analytical difficulties in this earlier work were resolved by C. C. Lin in his doctoral dissertation (Lin 1945), which formed the basis of the first monograph on hydrodynamic stability (Lin 1955).

The difficulties of experimentally reproducing the conditions of the linear stability theory for a boundary layer were overcome by Schubauer and Skramstad (1943), who obtained the waves predicted by Heisenberg and Tollmien (now known as *Tollmien–Schlichting waves*). Hans Liepmann (1997) recalls that:

> Sometime in 1941 with the war in Europe in its second year, I came up the narrow substandard stairs in the Guggenheim laboratory. At the top of the stairs I met Clark Millikan handing a sheaf of papers to von Kármán with the words "*It's a complete German victory!*" I was stunned. Fortunately, however, the victory was not another one won by Hitler but referred to the experimental verification of the [German] theory … by Schubauer and Skramstad… . I still vividly remember the impact these experiments had on me; they forcefully demonstrated … the beginning of a new area in transition research.

Drazin and Reid's monograph appeared as a successor to that of Lin (1955) in the same series. That the subject had grown is clear from the relative lengths (525 versus 155 pages). Lin concentrated on viscous shear flows, although he included brief chapters on circular Couette and convective motions. The latter subjects were subsequently treated more extensively by Chandrasekhar (1961), who, on the other hand, omitted viscous shear flows. Drazin and Reid cover the ground of both of these distinguished predecessors, giving somewhat more detail than Lin but rather less than Chandrasekhar. As they announce in their Preface, although hydrodynamic stability is important in engineering, meteorology, oceanography and astrophysics, their book 'is written from the point of view intrinsic to fluid mechanics and applied mathematics [and] emphasize[s] the analytical aspects of the theory'; however, 'wherever possible, [they] relate the theory to experimental and numerical results.'

The classical theory, which is thoroughly covered by Drazin & Reid, is essentially linear, but they also give a moderately extensive treatment of nonlinear stability, including dynamical-systems theory, Lorenz's (1963) seminal model of nonlinear convection, and strange attractors. They do not cover chaos, which (although still controversial in its application to open flows such as boundary layers) is now a field in itself and the subject of several monographs [e.g. Manneville (1990) and Holmes, Lumley, & Berkooz (1996)]. And, although they remark that 'spatial modes seem to describe weak instability of parallel flows more faithfully than temporal modes', they were too early to appreciate the importance of spatial instability and non-normal operators. Spatial instability, the hydrodynamic aspects of which are informed by corresponding developments in plasma physics, is covered by Huerre & Rossi (1998), and both spatial instability and non-normal operators are covered by Schmid & Henningson (2001).

The suggestion that transient disturbances might achieve very large values before decaying, and thus lead to transition at Reynolds numbers significantly smaller than those predicted by classical stability theory, goes back to Orr (1907), but it is only recently (in part because of the necessary computation) that this scenario has been quantitatively associated with the non-normality of the Orr–Sommerfeld operator and the corresponding non-orthogonality of its eigenfunctions. It appears that small, threedimensional disturbances of a smooth flow may be transiently amplified by factors of the order of 10^5 even though all of the eigenfunctions ultimately decay [Trefethen *et al.* (1993)]. This startline denouement provides a plausible explanation of the discrepancies between classical stability theory and observation. It also marks the end of an era in which hydrodynamic stability admits adequate coverage in a single volume. Drazin and Reid's monograph is the high-water mark of that era and remains an essential occupant of the book shelves of every student of hydrodynamic stability.

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PREFACE

For nearly a century now, hydrodynamic stability has been recognized as one of the central problems of fluid mechanics. It is concerned with when and how laminar flows break down, their subsequent development, and their eventual transition to turbulence. It has many applications in engineering, in meteorology and oceanography, and in astrophysics and geophysics. Some of these applications are mentioned, but the book is written from the point of view intrinsic to fluid mechanics and applied mathematics. Thus, although we have emphasized the analytical aspects of the theory, we have also tried, wherever possible, to relate the theory to experimental and numerical results.

Our aim in writing this book has been twofold. Firstly, in Chapters 1–4, to describe the fundamental ideas, methods, and results in three major areas of the subject: thermal convection, rotating and curved flows, and parallel shear flows. Secondly, to provide an introduction to some aspects of the subject which are of current research interest. These include some of the more recent developments in the asymptotic theory of the Orr–Sommerfeld equation in Chapter 5, some applications of the linear stability theory in Chapter 6 and finally, in Chapter 7, a discussion of some of the fundamental ideas involved in current work on the nonlinear theory of hydrodynamic stability.

Each chapter ends with a number of problems which often extend or supplement the main text as well as provide exercises to help the reader understand the topics. An asterisk is used to indicate those problems which we judge to be relatively long or difficult. Some hints and references are given to help in the solution of many of the problems. We have also prepared answers to the problems.

Thus this is a textbook suitable for a graduate course on the fundamental ideas and methods and on the major applications of the theory of hydrodynamic stability. It also leads the reader up to the frontiers of research on selected topics. In general we have assumed that the reader is familiar with whatever mathematical methods are needed, notably in the theories of ordinary and of partial differential equations and in the theory of functions of a complex variable. But we have explained some specialized and modern mathematical points at length where it seems that they are likely to be unfamiliar to most readers.

We are grateful to our many colleagues throughout the world who have responded so generously to our various inquiries. In particular, we thank A. Davey, T. H. Hughes, and L. M. Mack for providing new or unpublished numerical results, R. J. Donnelly, E. L. Koschmieder and S. A. Thorpe for providing photographs, J. P. Cleave for advice on some mathematical points, L. C. Woods for advice on the presentation of the material, B. S. Ng for detailed comments on Chapters 1–5, and A. Davey and J. T. Stuart for constructive criticisms of a draft of Chapter 7. For help with the typing of the manuscript we also thank N. Thorp in Bristol and M. Bowie, F. Flowers, L. Henley, and M. Newman in Chicago. We are especially indebted to S. Chandrasekhar and C. C. Lin, who have contributed so much to the theory of hydrodynamic stability; through their papers and books, and through our personal contacts with them, they have greatly influenced our work on the subject. One of us (W.H.R.) also wishes to acknowledge with thanks the generous support provided over the years by the U.S. National Science Foundation, most recently under grant no. MCS 78–01249.

And, finally, we should like to thank G. K. Batchelor not only for his help as editor of this series but also for his kindness during an early stage in our careers when it was our good fortune to be associated with him.

Bristol P.G.D.
Chicago W.H.R.
August 1979

My collaboration with Philip Drazin began when he and Judith Drazin spent a sabbatical year at the University of Chicago. It was a great year indeed and our discussions at that time eventually led to the writing of *Hydrodynamic Stability*. The actual writing, however, took much longer than either of us would have wished and I am grateful to Philip for his endless patience throughout. In short it was a long but wonderful collaboration.

Indianapolis W.H.R.
March 2003