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P. G. Drazin and W. H. Reid
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HYDRODYNAMIC STABILITY

SECOND EDITION

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TO OUR PARENTS

Isaac and Leah Drazin

William and Edna Reid

CONTENTS

	<i>page</i>
<i>Foreword by John Miles</i>	xiii
<i>Preface</i>	xix
1 INTRODUCTION	
1 Introduction	1
2 Mechanisms of instability	4
3 Fundamental concepts of hydrodynamic stability	8
4 Kelvin–Helmholtz instability	14
5 Break-up of a liquid jet in air	22
Problems for chapter 1	27
2 THERMAL INSTABILITY	
6 Introduction	32
7 The equations of motion	34
The exact equations, 34; The Boussinesq equations, 35	
8 The stability problem	37
The linearized equations, 37; The boundary conditions, 40; Normal modes, 42	
9 General stability characteristics	44
Exchange of stabilities, 44; A variational principle, 45	
10 Particular stability characteristics	50
Free–free boundaries, 50; Rigid–rigid boundaries, 51; free–rigid boundaries, 52	
11 The cells	52
12 Experimental results	59
13 Some applications	62
Problems for chapter 2	63

3 CENTRIFUGAL INSTABILITY

14	Introduction	69
15	Instability of an inviscid fluid	71
	Three-dimensional disturbances, 73; Axisymmetric disturbances, 77; Two-dimensional disturbances, 80	
16	Instability of Couette flow of an inviscid fluid	82
17	The Taylor problem	88
	Axisymmetric disturbances, 90; Two-dimensional disturbances, 103; Three-dimensional disturbances, 104; Some experimental results, 104	
18	The Dean problem	108
	The Dean problem, 108; The Taylor–Dean problem, 113	
19	The Görtler problem	116
	Problems for chapter 3	121

4 PARALLEL SHEAR FLOWS

20	Introduction	124
	The inviscid theory	
21	The governing equations	126
22	General criteria for instability	131
23	Flows with piecewise-linear velocity profiles	144
	Unbounded vortex sheet, 145; Unbounded shear layer, 146; Bounded shear layer, 147	
24	The initial-value problem	147
	The viscous theory	
25	The governing equations	153
26	The eigenvalue spectrum for small Reynolds numbers	158
	A perturbation expansion, 159; Sufficient conditions for stability, 161	
27	Heuristic methods of approximation	164
	The reduced equation and the inviscid approximations, 165; The boundary-layer approximation near a rigid wall, 167; The WKBJ approximations, 167; The local turning-point approximations,	

CONTENTS

ix

	171; The truncated equation and Tollmien's improved viscous approximations, 175; The viscous correction to the singular inviscid solution, 177	
28	Approximations to the eigenvalue relation Symmetrical flows in a channel, 181; Flows of the boundary-layer type, 183; The boundary-layer approximation to $\phi_3(z)$, 184; The WKBJ approximation to $\phi_3(z)$, 185; The local turning-point approximation to $\phi_3(z)$, 188; Tollmien's improved approximation to $\phi_3(z)$, 191	180
29	The long-wave approximation for unbounded flows	196
30	Numerical methods of solution Expansions in orthogonal functions, 203; Finite-difference methods, 206; Initial-value methods (shooting), 207	202
31	Stability characteristics of various basic flows Plane Couette flow, 212; Poiseuille flow in a circular pipe, 216; Plane Poiseuille flow, 221; Combined plane Couette and plane Poiseuille flow, 223; The Blasius boundary-layer profile, 224; The asymptotic suction boundary-layer profile, 227; Boundary layers at separation, 229; The Falkner–Skan profiles, 231; The Bickley jet, 233; The hyperbolic-tangent shear layer, 237	211
32	Experimental results	239
	Problems for chapter 4	245
	5 UNIFORM ASYMPTOTIC APPROXIMATIONS	
33	Introduction Plane Couette flow	251
34	The integral representations of the solutions	256
35	The differential equation method General velocity profiles	263
36	A preliminary transformation	265

x	CONTENTS	
37	The inner and outer expansions	267
	The inner expansions, 268; The outer expansions, 271; The central matching problem, 276; Composite approximations, 278	
38	Uniform approximations	280
	The solution of well-balanced type, 280; The solutions of balanced type, 280; The solutions of dominant-recessive type, 283	
39	A comparison with Lin's theory	285
40	Preliminary simplification of the eigenvalue relation	290
41	The uniform approximation to the eigenvalue relation	295
	A computational form of the first approximation to the eigenvalue relation, 299; Results for plane Poiseuille flow, 301	
42	A comparison with the heuristic approximations to the eigenvalue relation	305
	The local turning-point approximation to $\phi_3(z)$, 305; Tollmien's improved approximation to $\phi_3(z)$, 306; The uniform approximation to $\phi_3(z)$ based on the truncated equation, 308; The uniform approximation to $\phi_3(z)$ based on the Orr–Sommerfeld equation, 309	
43	A numerical treatment of the Orr–Sommerfeld problem using compound matrices	311
	Symmetrical flows in a channel, 315; Boundary-layer flows, 316	
	Problems for chapter 5	317
	6 ADDITIONAL TOPICS IN LINEAR STABILITY THEORY	
44	Instability of parallel flow of a stratified fluid	320
	Introduction, 320; Internal gravity waves and Rayleigh–Taylor instability, 324; Kelvin–Helmholtz instability, 325	
45	Baroclinic instability	333
46	Instability of the pinch	339
47	Development of linear instability in time and space	345
	Initial-value problems, 345; Spatially growing modes, 349	

CONTENTS xi

48	Instability of unsteady flows	353
	Introduction, 353; Instability of periodic flows, 354; Instability of other unsteady basic flows, 361	
	Problems for chapter 6	363

7 NONLINEAR STABILITY

49	Introduction	370
	Landau's theory, 370; Discussion, 376	
50	The derivation of ordinary differential systems governing stability	380
51	Resonant wave interactions	387
	Internal resonance of a double pendulum, 387; Resonant wave interactions, 392	
52	Fundamental concepts of nonlinear stability	398
	Introduction to ordinary differential equations, 398; Introduction to bifurcation theory, 402; Structural stability, 407; Spatial development of nonlinear stability, 416; Critical layers in parallel flow, 420	
53	Additional fundamental concepts of nonlinear stability	423
	The energy method, 424; Maximum and minimum energy in vortex motion, 432; Application of bound- ary-layer theory to cellular instability, 434	
54	Some applications of the nonlinear theory	435
	Bénard convection, 435; Couette flow, 442; Parallel shear flows, 450	
	Problems for chapter 7	458

APPENDIX. A CLASS OF
GENERALIZED AIRY FUNCTIONS

A1	The Airy functions $A_k(z)$	465
A2	The functions $A_k(z, p)$, $B_0(z, p)$ and $B_k(z, p)$	466
A3	The functions $A_k(z, p, q)$ and $B_k(z, p, q)$	472
A4	The zeros of $A_1(z, p)$	477
	Addendum: Weakly non-parallel theories for the Blasius boundary layer	479
	<i>Solutions</i>	481
	<i>Bibliography and author index</i>	559
	<i>Motion picture index</i>	595
	<i>Subject index</i>	597

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Frontmatter

[More information](#)

FOREWORD

The study of hydrodynamic stability goes back to the theoretical work of Helmholtz (1868), Kelvin (1871) and Rayleigh (1879, 1880) on inviscid flows and, above all, the experimental investigations of Reynolds (1883), which initiated the systematic study of viscous shear flows. Reynolds's work stimulated the theoretical investigations of Orr (1907) and Sommerfeld (1908), who independently considered small, traveling-wave disturbances of an otherwise steady, parallel flow and derived (what is now known as) the Orr–Sommerfeld equation.

Early attempts to solve the Orr–Sommerfeld equation for the flow associated with the uniform, relative motion of two parallel plates (plane Couette flow) led to the prediction of stability for all Reynolds numbers, in apparent disagreement with experiment (although the prediction *is* correct for infinitesimal disturbances). G. I. Taylor (1923), referring to this work, remarked that:

This problem has been chosen because it seemed probable that the mathematical analysis might prove comparatively simple; but ... it has actually proved very complicated and difficult. [Moreover] it would be extremely difficult to verify experimentally any conclusions which might be arrived at in this case, because of the difficulty of designing apparatus in which the required boundary conditions are approximately satisfied.

It is very much easier to design apparatus for studying the flow of fluid under pressure through a tube, or the flow between two concentric rotating cylinders. The experiments of Reynolds and others suggest that [for] flow through a circular tube, infinitely small disturbances are stable, while larger disturbances increase provided the speed of flow is greater than a certain amount. The study of the fluid stability when the disturbances are not ... infinitely small is extremely difficult. It seems more promising therefore to examine the stability of liquid contained between concentric rotating cylinders [circular Couette flow].

Taylor then proceeded to carry out both stability calculations and experiments for circular Couette flow and concluded with the characteristic statement that ‘The accuracy with which the observed and calculated sets of points [agree] is remarkable when it is remembered how complicated was the analysis employed in obtaining them.’

As Taylor anticipated, corresponding results for parallel shear flows, for which the Reynolds number at transition is relatively large, proved to be more elusive. An essential difficulty in understanding the instability of such flows is the dual role of viscosity, which may be either stabilizing, as intuitively expected, or destabilizing, a phenomenon adumbrated by Taylor (1915) and explicated by Prandtl (1921) and Tollmien (1929). Moreover, the asymptotic (Reynolds number $\rightarrow \infty$) methods invoked by Heisenberg (1924), Tollmien (1929) and Schlichting (1933) to solve the Orr–Sommerfeld equation were heuristic and left doubts about the theoretical predictions, especially as these predictions conflicted with observation. The analytical difficulties in this earlier work were resolved by C. C. Lin in his doctoral dissertation (Lin 1945), which formed the basis of the first monograph on hydrodynamic stability (Lin 1955).

The difficulties of experimentally reproducing the conditions of the linear stability theory for a boundary layer were overcome by Schubauer and Skramstad (1943), who obtained the waves predicted by Heisenberg and Tollmien (now known as *Tollmien–Schlichting waves*). Hans Liepmann (1997) recalls that:

Sometime in 1941 with the war in Europe in its second year, I came up the narrow substandard stairs in the Guggenheim laboratory. At the top of the stairs I met Clark Millikan handing a sheaf of papers to von Kármán with the words “*It’s a complete German victory!*” I was stunned. Fortunately, however, the victory was not another one won by Hitler but referred to the experimental verification of the [German] theory ... by Schubauer and Skramstad... I still vividly remember the impact these experiments had on me; they forcefully demonstrated ... the beginning of a new area in transition research.

Drazin and Reid’s monograph appeared as a successor to that of Lin (1955) in the same series. That the subject had grown is clear

from the relative lengths (525 versus 155 pages). Lin concentrated on viscous shear flows, although he included brief chapters on circular Couette and convective motions. The latter subjects were subsequently treated more extensively by Chandrasekhar (1961), who, on the other hand, omitted viscous shear flows. Drazin and Reid cover the ground of both of these distinguished predecessors, giving somewhat more detail than Lin but rather less than Chandrasekhar. As they announce in their Preface, although hydrodynamic stability is important in engineering, meteorology, oceanography and astrophysics, their book 'is written from the point of view intrinsic to fluid mechanics and applied mathematics [and] emphasize[s] the analytical aspects of the theory'; however, 'wherever possible, [they] relate the theory to experimental and numerical results.'

The classical theory, which is thoroughly covered by Drazin & Reid, is essentially linear, but they also give a moderately extensive treatment of nonlinear stability, including dynamical-systems theory, Lorenz's (1963) seminal model of nonlinear convection, and strange attractors. They do not cover chaos, which (although still controversial in its application to open flows such as boundary layers) is now a field in itself and the subject of several monographs [e.g. Manneville (1990) and Holmes, Lumley, & Berkooz (1996)]. And, although they remark that 'spatial modes seem to describe weak instability of parallel flows more faithfully than temporal modes', they were too early to appreciate the importance of spatial instability and non-normal operators. Spatial instability, the hydrodynamic aspects of which are informed by corresponding developments in plasma physics, is covered by Huerre & Rossi (1998), and both spatial instability and non-normal operators are covered by Schmid & Henningson (2001).

The suggestion that transient disturbances might achieve very large values before decaying, and thus lead to transition at Reynolds numbers significantly smaller than those predicted by classical stability theory, goes back to Orr (1907), but it is only recently (in part because of the necessary computation) that this scenario has been quantitatively associated with the non-normality of the Orr–Sommerfeld operator and the corresponding non-orthogonality of its eigenfunctions. It appears that small, three-

dimensional disturbances of a smooth flow may be transiently amplified by factors of the order of 10^5 even though all of the eigenfunctions ultimately decay [Trefethen *et al.* (1993)]. This startline denouement provides a plausible explanation of the discrepancies between classical stability theory and observation. It also marks the end of an era in which hydrodynamic stability admits adequate coverage in a single volume. Drazin and Reid's monograph is the high-water mark of that era and remains an essential occupant of the book shelves of every student of hydrodynamic stability.

John Miles
 University of California
 San Diego

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FOREWORD

xvii

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PREFACE

For nearly a century now, hydrodynamic stability has been recognized as one of the central problems of fluid mechanics. It is concerned with when and how laminar flows break down, their subsequent development, and their eventual transition to turbulence. It has many applications in engineering, in meteorology and oceanography, and in astrophysics and geophysics. Some of these applications are mentioned, but the book is written from the point of view intrinsic to fluid mechanics and applied mathematics. Thus, although we have emphasized the analytical aspects of the theory, we have also tried, wherever possible, to relate the theory to experimental and numerical results.

Our aim in writing this book has been twofold. Firstly, in Chapters 1–4, to describe the fundamental ideas, methods, and results in three major areas of the subject: thermal convection, rotating and curved flows, and parallel shear flows. Secondly, to provide an introduction to some aspects of the subject which are of current research interest. These include some of the more recent developments in the asymptotic theory of the Orr–Sommerfeld equation in Chapter 5, some applications of the linear stability theory in Chapter 6 and finally, in Chapter 7, a discussion of some of the fundamental ideas involved in current work on the nonlinear theory of hydrodynamic stability.

Each chapter ends with a number of problems which often extend or supplement the main text as well as provide exercises to help the reader understand the topics. An asterisk is used to indicate those problems which we judge to be relatively long or difficult. Some hints and references are given to help in the solution of many of the problems. We have also prepared answers to the problems.

Thus this is a textbook suitable for a graduate course on the fundamental ideas and methods and on the major applications of

the theory of hydrodynamic stability. It also leads the reader up to the frontiers of research on selected topics. In general we have assumed that the reader is familiar with whatever mathematical methods are needed, notably in the theories of ordinary and of partial differential equations and in the theory of functions of a complex variable. But we have explained some specialized and modern mathematical points at length where it seems that they are likely to be unfamiliar to most readers.

We are grateful to our many colleagues throughout the world who have responded so generously to our various inquiries. In particular, we thank A. Davey, T. H. Hughes, and L. M. Mack for providing new or unpublished numerical results, R. J. Donnelly, E. L. Koschmieder and S. A. Thorpe for providing photographs, J. P. Cleave for advice on some mathematical points, L. C. Woods for advice on the presentation of the material, B. S. Ng for detailed comments on Chapters 1–5, and A. Davey and J. T. Stuart for constructive criticisms of a draft of Chapter 7. For help with the typing of the manuscript we also thank N. Thorp in Bristol and M. Bowie, F. Flowers, L. Henley, and M. Newman in Chicago. We are especially indebted to S. Chandrasekhar and C. C. Lin, who have contributed so much to the theory of hydrodynamic stability; through their papers and books, and through our personal contacts with them, they have greatly influenced our work on the subject. One of us (W.H.R.) also wishes to acknowledge with thanks the generous support provided over the years by the U.S. National Science Foundation, most recently under grant no. MCS 78–01249.

And, finally, we should like to thank G. K. Batchelor not only for his help as editor of this series but also for his kindness during an early stage in our careers when it was our good fortune to be associated with him.

Bristol
 Chicago
 August 1979

P.G.D.
 W.H.R.

My collaboration with Philip Drazin began when he and Judith Drazin spent a sabbatical year at the University of Chicago. It was a great year indeed and our discussions at that time eventually led to the writing of *Hydrodynamic Stability*. The actual writing, however, took much longer than either of us would have wished and I am grateful to Philip for his endless patience throughout. In short it was a long but wonderful collaboration.

Indianapolis
 March 2003

W.H.R.