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CHAPTER 1

INTRODUCTION

Yet not every solution of the equations of motion, even if it is exact, can actually occur in Nature. The flows that occur in Nature must not only obey the equations of fluid dynamics, but also be stable.
– L. D. Landau & E. M. Lifshitz (1959)

1 Introduction

The essential problems of hydrodynamic stability were recognized and formulated in the nineteenth century, notably by Helmholtz, Kelvin, Rayleigh and Reynolds. It is difficult to introduce these problems more clearly than in Osborne Reynolds's (1883) own description of his classic series of experiments on the instability of flow in a pipe.

The ... experiments were made on three tubes .... The diameters of these were nearly 1 inch, $\frac{1}{2}$ inch and $\frac{1}{4}$ inch. They were all ... fitted with trumpet mouthpieces, so that the water might enter without disturbance. The water was drawn through the tubes out of a large glass tank, in which the tubes were immersed, arrangements being made so that a streak or streaks of highly coloured water entered the tubes with the clear water.

The general results were as follows:–

(1) When the velocities were sufficiently low, the streak of colour extended in a beautiful straight line through the tube, Fig. 1.1(*a*).

(2) If the water in the tank had not quite settled to rest, at sufficiently low velocities, the streak would shift about the tube, but there was no appearance of sinuosity.

(3) As the velocity was increased by small stages, at some point in the tube, always at a considerable distance from the trumpet or intake, the colour band would all at once mix up with the surrounding water, and fill the rest of the tube with a mass of coloured water, as in Fig. 1.1(*b*). Any increase in the velocity caused the point of break down to approach the trumpet, but with no velocities that were tried did it reach this. On viewing the tube by the light of an electric spark, the mass of colour resolved itself into a mass of more or less distinct curls, showing eddies, as in Fig. 1.1(*c*).

Reynolds went on to show that the *laminar flow*, the smooth flow he described in paragraph (1), breaks down when Va/ν exceeds a certain critical value, V being the maximum velocity of the water in the tube, a the radius of the tube, and ν the kinematic viscosity of

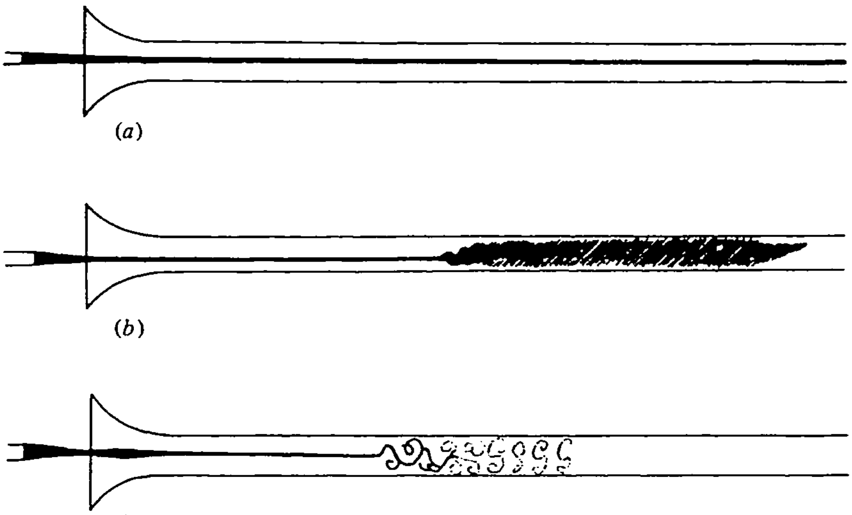

Fig. 1.1. (*a*) Laminar flow in a pipe. (*b*) Transition to turbulent flow in a pipe. (*c*) Transition to turbulent flow as seen when illuminated by a spark. (From Reynolds 1883.)

water at the appropriate temperature. This dimensionless number Va/ν, now called the *Reynolds number*, specifies any class of dynamically similar flows through a pipe; here we shall denote the number by R. The series of experiments gave the critical value R_c of the Reynolds number as nearly 13 000. However,

> the critical velocity was very sensitive to disturbance in the water before entering the tubes . . . . This at once suggested the idea that the condition might be one of instability for disturbance of a certain magnitude and stable for smaller disturbances.

At the critical velocity

> another phenomenon . . . was the intermittent character of the disturbance. The disturbance would suddenly come on through a certain length of tube and pass away and then come on again, giving the appearance of flashes, and these flashes would often commence successively at one point in the pipe. The appearance when the flashes succeeded each other rapidly was as shown . . .

in Fig. 1.2. Such 'flashes' are now called *turbulent spots* or *turbulent bursts*. Below the critical Reynolds number there was laminar Poiseuille flow with a parabolic velocity profile, the resistance of the

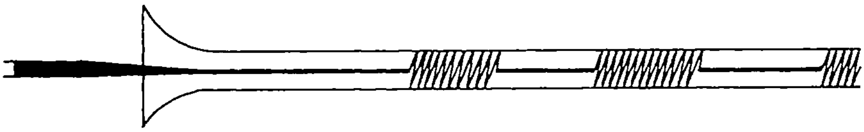

Fig. 1.2. Turbulent spots in a pipe. (From Reynolds 1883.)

pipe to the flow of water being proportional to the mean velocity. As the velocity increased above its critical value, Reynolds found that the flow became *turbulent*, with a chaotic motion that strongly diffused the dye throughout the water in the tube. The resistance of the pipe to turbulent flow grew in proportion to the square of the mean velocity.

Later experimentalists have introduced disturbances of finite amplitude at the intake or used tubes with roughened walls to find R_c as low as 2000, and have used such regular flows and such smooth-walled tubes that R_c was 40 000 or even more.

Reynolds's description illustrates the aims of the study of hydrodynamic stability: to find whether a given laminar flow is unstable and, if so, to find how it breaks down into turbulence or some other laminar flow.

Methods of analysing the stability of flows were formulated in Reynolds's time. The method of normal modes for studying the oscillations and instability of a dynamical system of particles and rigid bodies was already highly developed. A known solution of Newton's or Lagrange's equations of motion for the system was perturbed. The equations were linearized by neglecting products of the perturbations. It was further assumed that the perturbation of each quantity could be resolved into independent components or modes varying with time t like e^{st} for some constant s, which is in general complex. The values of s for the modes were calculated from the linearized equations. If the real part of s was found to be positive for any mode, the system was deemed unstable because a general initial small perturbation of the system would grow exponentially in time until it was no longer small. Stokes, Kelvin and Rayleigh adapted this method of normal modes to fluid dynamics. The essential mathematical difference between fluid and particle dynamics is that the equations of motion are partial rather than ordinary differential equations. This difference leads to many technical difficulties in hydrodynamic stability, which, to this day, have been overcome for only a few classes of flows with very simple configurations.

Indeed, Reynolds's experiment itself is still imperfectly understood. However, we can explain qualitatively the transition from laminar flow to turbulence with some confidence. Poiseuille flow with a parabolic profile is stable to infinitesimal disturbances at all Reynolds numbers. Some way below the observed critical Reynolds number a finite disturbance may grow if it is not too small. Above the critical Reynolds number quite small disturbances, perhaps introduced at the inlet or by an irregularity of the wall of the tube, grow rapidly with a sinuous motion. Soon they grow so much that nonlinearity becomes strong and large eddies (Fig. 1.1(*c*)) or turbulent spots (Fig. 1.2) form. At high Reynolds numbers turbulence ensues at once and the flow becomes random and strongly nonlinear everywhere.† This instability of Poiseuille flow may be contrasted with that of plane Poiseuille flow, which is unstable to infinitesimal disturbances at sufficiently large values of the Reynolds number. This explanation is supported by the treatment of the theory of the linear stability of Poiseuille flow in § 31 and of the nonlinear stability in § 49.1.

The physical mechanisms of Reynolds's experiments on instability of Poiseuille flow in a pipe are vividly illustrated by a film loop made by Stewart (FL 1968) for the Educational Development Center. This loop consists of edited excerpts from his longer film on *Turbulence* (Stewart F 1968). Details of these and other motion pictures on hydrodynamic stability may be found at the end of the bibliography.

2 Mechanisms of instability

Few laminar flows correspond to known solutions of the nonlinear equations of motion. Fewer are simple enough to allow detailed analysis of their instability. Consequently research on hydrodynamic stability has been deep but narrow. We are forced to study

† Many of the features of the transition from laminar to turbulent flow can easily be seen by observing the smoke from a cigarette. Light the cigarette, point the burning tip upwards in still air, and observe the smoke as it rises from rest at the hot tip.

the stability of a few classes of simple laminar flows, mostly with planar, axial or spherical symmetry. Unfortunately, their simplicity obscures some general aspects of instability, especially three-dimensional ones, such as flow with stretching of vortex lines. To gain a wider understanding of hydrodynamic stability and to put these simple flows in perspective as prototypes, it is helpful to sketch the important physical mechanisms of instability.

Broadly speaking, one may say that instability occurs because there is some disturbance of the equilibrium of the external forces, inertia and viscous stresses of a fluid. We shall discuss the external forces first. External forces of interest are buoyancy in a fluid of variable density, surface tension, magnetohydrodynamic forces, etc. It is also convenient to regard centrifugal and Coriolis forces as external forces when there is rotation of the whole system in which the fluid moves. If heavy fluid rests above light fluid it is clear that the fluids tend to overturn under the action of gravity. A similar instability occurs on the free surface of a container of liquid when it is moved downwards at a uniform acceleration greater than the gravitational acceleration. There is in fact a close analogy between the problem of instability of a fluid of variable density, namely Bénard convection bounded by horizontal planes, and the problem of instability of axisymmetric swirling flow of homogeneous fluid, namely Taylor vortices bounded by two coaxial rotating cylinders. The analogue of the density turns out to be the square of the circulation or swirl. If in Couette flow the circulation around the inner cylinder is greater in magnitude than that around the outer, the centrifugal force tends to throw out the fluid near the inner cylinder as an overturning instability. This centrifugal instability may occur also in flows along a rigid curved surface such as a concave wall of a channel. Surface tension resists the increase of area of a surface and so exerts a stabilizing influence, particularly on disturbances of small length scale. A magnetic field can inhibit the motion of an electrically conducting fluid across the magnetic lines of force and thereby usually tends to stabilize flows.

In the absence of any external force or of viscosity, a fluid moves according to the equilibrium between its inertia and internal stresses of pressure. A small disturbance may upset this equilibrium. The tendency of fluid to move down pressure gradients can be seen to amplify disturbances of certain flows and thereby create instability. This instability can be depicted more reliably in terms of interactions of the vortex lines, which are convected and stretched by the motion of the fluid.

An obvious effect of viscosity is to dissipate the energy of any disturbance and thereby stabilize a flow. Indeed, for this reason any bounded flow is stable if the viscosity is large enough. So, by and large, viscosity has a stabilizing influence. Viscosity has also the more complicated effect of diffusing momentum. This can make some flows, notably parallel shear flows, *un*stable although the same flows of an inviscid fluid are stable.

Thermal conductivity, or molecular diffusion of heat, has some effects similar to those of viscosity, or molecular diffusion of momentum. It tends to smooth out the temperature differences of a disturbance and is usually a stabilizing influence.

It is natural to consider the stability of primarily steady flows, but unsteady ones are also of some practical importance. The acceleration of a laminar flow plays an identifiable role in its stability. Analysis is difficult in general, but it emerges that acceleration of a laminar flow has a stabilizing and deceleration a destabilizing tendency. Flows that oscillate in time, such as Poiseuille flow driven through a circular pipe by an oscillatory pressure gradient, have intricate stability characteristics. Parametric stability or instability may occur, whereby the free oscillations of disturbances of the mean flow resonate with the forced oscillations of the flow.

Finally, the boundaries of a flow are an important factor. They constrain the development of a disturbance and usually the closer they are together the more stable the flow. However, they sometimes give rise to strong shear in boundary layers which is diffused outwards by viscosity and so leads to instability of the flow.

In a typical flow more than one of these mechanisms may act. For example, in plane Poiseuille flow the dual effects of viscosity, the inertia and the boundaries all influence the instability. Plane Poiseuille flow of inviscid fluid is stable. At large but finite Reynolds numbers the diffusion of momentum from thin shear layers near the walls leads to instability. At small Reynolds numbers the dissipative role of viscosity is dominant and there is stability. This leads to a critical value of the dimensionless number R representing the ratio of the magnitudes of the destabilizing forces of shear and stabilizing viscous forces for which their effects may be said to balance.

This summary of mechanisms of hydrodynamic stability will be given substance by detailed problems in this and later chapters. But the necessary details of the instability of any particular flow should not obscure these general mechanisms, whose recognition helps to classify as well as to understand problems. Instability arising from an upset of the equilibrium between an external force and dissipative effects is usually simpler than inertial instability. Prototype problems of this simpler instability will be analysed first, the linear instability of a fluid heated from below in Chapter 2 and centrifugal instability in Chapter 3. If fluid lies at rest between two horizontal plates, the lower one being hotter than the upper, then we have light fluid below heavy fluid. The buoyancy tends to overturn the fluid. This tendency is countered by the dissipative and diffusive effects of viscosity and thermal conductivity. The dimensionless number typifying the ratio of the destabilizing buoyancy to the stabilizing diffusive forces is called the *Rayleigh number*; its critical value is calculated and related to experiments. In this flow, viscosity and thermal conductivity have only stabilizing effects. The instability of Couette flow between rotating cylinders is analogous, viscosity tending only to stabilize the centrifugal instability. The linear instability of parallel shear flows is treated in Chapter 4, where it is shown that viscosity typically plays the dual roles of stabilizer and destabilizer. There is an imbalance between the inertia and both the dissipative and diffusive effects of viscosity. The physical mechanism of this instability and its mathematical description is more difficult to understand than that of instability due to an external force, and will be explained in detail in Chapter 4. The most difficult mathematical topics, involving the asymptotic theory of the solutions of the Orr–Sommerfeld equation which governs the problem, will be elaborated in Chapter 5. Chapter 5 may be ignored by the less mathematically inclined reader who is interested in stability characteristics rather than in deducing them. More complicated flows whose stability is governed by more than two of the mechanisms will be presented in Chapter 6. Their mathematical difficulties are so formidable that only a few illustrative examples will be given. Also a few other problems will be discussed, in particular the instability of unsteady basic flows. Chapters 2–6 treat almost entirely linear theory, which is now well formed and understood. The important topic of nonlinear instability is introduced in Chapter 7. Nonlinear theory has not reached a definitive stage yet, knowledge of the transition from laminar flow to turbulence being far from complete. Research has intensified in the last two decades, however, and has led to an increased understanding of transition beyond the stage of linear instability.

To prepare for this course of detailed analysis, this chapter ends with an account of the concepts of stability and of the methods of linear stability, including some classic problems as illustrations.

3 Fundamental concepts of hydrodynamic stability

To analyse the stability of any laminar flow one must first find the velocity $\mathbf{U}(\mathbf{x}, t)$ and other fields, such as pressure $P(\mathbf{x}, t)$ and temperature $\Theta(\mathbf{x}, t)$, needed to specify the laminar flow at each point $\mathbf{x}$ and time t. These fields define the *basic flow*. The fields may be steady or unsteady, and should satisfy the appropriate equations of motion and boundary conditions. Choice of suitable equations to model an observed flow and solution of the equations are often difficult tasks, but we shall suppose here that the equations and their solution are completely known, even though minor features of the observed flow may be neglected or only an approximate solution found.

Physically we want to know whether the basic flow can be observed or not. If it is disturbed slightly, the disturbance may either die away, persist as a disturbance of similar magnitude or grow so much that the basic flow becomes a different laminar or a turbulent flow. Broadly speaking, we call such disturbances (*asymptotically*) *stable*, *neutrally stable* or *unstable* respectively. All possible slight disturbances are likely to be excited in some degree by small irregularities or vibrations of the basic flow in practice, so it will persist only if it is stable to *all* slight disturbances. In seeking more precise definitions of stability we may be guided by the considerable mathematical literature of stability of solutions of ordinary and partial differential equations, but must frame the definitions to further our physical understanding. The choice of useful definitions of ‘disturbed slightly’, ‘die away’ and ‘disturbance of similar magnitude' is usually clear unless the basic flow is unsteady or nonlinearity is significant. Definitions for these two cases are still controversial, and will be discussed in § 48 and Chapter 7. But at the outset we should recognize that the important thing is to understand how disturbances evolve in time rather than to argue about the definition of their stability.

It may help the mathematically inclined reader to formalize these definitions. However, the physically inclined reader may ignore this paragraph because in most applications common sense makes a formal definition unnecessary. Following the theory of stability of systems of ordinary differential equations, we say a basic flow is *stable* (*in the sense of Liapounov*) if, for any $\varepsilon > 0$, there exists some positive number δ (depending upon ε) such that if

$$\|\mathbf{u}(\mathbf{x}, 0) - \mathbf{U}(\mathbf{x}, 0)\|, \|p(\mathbf{x}, 0) - P(\mathbf{x}, 0)\|, \text{ etc. } < \delta,$$

then

$$\|\mathbf{u}(\mathbf{x}, t) - \mathbf{U}(\mathbf{x}, t)\|, \|p(\mathbf{x}, t) - P(\mathbf{x}, t)\|, \text{ etc.} < \varepsilon \quad \text{for all } t \geq 0,$$

where $\mathbf{u}$ is the velocity field and p is the pressure field, which satisfy the equations of motion and the boundary conditions. This definition means that the flow is stable if the perturbation is small for all time provided it is small initially, or, in yet other words, if the solution is uniformly continuous for all time with respect to the initial conditions. The precise meaning of 'small' or 'continuous' perturbation has to be assigned by definition of the positive definite norm. There is some freedom here, but it is physically useful to choose

$$\|\mathbf{u}(\mathbf{x}, t) - \mathbf{U}(\mathbf{x}, t)\| = \max_{\mathbf{x} \in \mathcal{V}} |\mathbf{u}(\mathbf{x}, t) - \mathbf{U}(\mathbf{x}, t)|$$

or

$$\left\{ \int_{\mathcal{V}} \tfrac{1}{2}\rho (\mathbf{u} - \mathbf{U})^2 \, d\mathbf{x} \right\}^{1/2} \text{ or } \left\{ \int_{\mathcal{V}} [\{\mathcal{V}^{1/3} \nabla (\mathbf{u} - \mathbf{U})\}^2 + (\mathbf{u} - \mathbf{U})^2] \, d\mathbf{x} \right\}^{1/2},$$

for example, where $\mathcal{V}$ is the domain of flow and ρ is the density of the fluid. Similarly, we may say the basic flow is *asymptotically stable* (*in the sense of Liapounov*) if moreover

$$\|\mathbf{u}(\mathbf{x}, t) - \mathbf{U}(\mathbf{x}, t)\|, \text{ etc. } \to 0 \quad \text{as} \quad t \to +\infty.$$

These definitions may be unsatisfactory when the norm of the basic flow itself decreases or increases substantially in time. Then a time-dependent norm may have to be carefully chosen to represent what the experimentalist or observer means intuitively by stability.

Other perturbations that might lead to instability arise from small changes in the boundary conditions due to irregularities in Nature or imperfections of laboratory equipment. The mathematical treatment of these perturbations is closely related to that of a small initial disturbance of the basic flow.

Also, it must be recognized that an unstable basic flow free of any disturbance cannot instantaneously be set up in the laboratory or arise in Nature. Rather a stable basic flow evolves in space or time until it becomes unstable, and the nature of the instability may be affected by the means of evolution.

Here we consider steady basic flows and *assume* that the equations of motion and the boundary conditions may be linearized for sufficiently small disturbances. Linearization is straightforward in principle and practice: products of the increments $\mathbf{u}'(\mathbf{x}, t) = \mathbf{u} - \mathbf{U}$, $p'(\mathbf{x}, t) = p - P$, etc., that is, of the total velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, t)$, etc. of the disturbed flow less their respective values for the basic flow, are neglected. Thereby a linear homogeneous system of partial differential equations and boundary conditions is obtained. These have coefficients that may vary in space but not time because the basic flow is steady. Experience with the method of separation of variables and with Laplace transforms suggests that in general the solutions of the system can be expressed as the real parts of integrals of components, each component varying with time like e^{st} for some complex number $s = \sigma + i\omega$. The linear system will determine the values of s and the spatial variation of corresponding components as eigenvalues and eigenfunctions.

If the basic flow has some simple symmetry, the linear system may be transformed with respect to some of the space variables as well as time. For example, Poiseuille flow has basic velocity and pressure respectively given by

$$\mathbf{U} = V(1 - r^2/a^2)\mathbf{i},$$
$$P = p_0 - 4\rho\nu Vx/a^2 \quad \text{for } 0 \leqslant r \leqslant a,\ 0 \leqslant \theta < 2\pi,\ -\infty < x < \infty,$$

where ρ is the density of the fluid, cylindrical polar coordinates (x, r, θ) are used, and $\mathbf{i}$ is the unit vector parallel to the x-axis. The axial symmetry of this flow is such that the coefficients of the