PART I

Integral Geometry in the Plane

CHAPTER 1

Convex Sets in the Plane

1. Introduction

Convex sets play an important role in integral geometry. For this reason we will review here their principal properties, especially those which will be needed in the following sections. In this chapter we consider convex sets in the plane. For convex sets in n-dimensional euclidean space, see Chapter 13. For a more complete treatment, refer to the classical books of Blaschke [50] and Bonnesen and Fenchel [63], or to the more modern texts of Benson [27], Eggleston [162], Grünbaum [247], Jaglom and Boltjanski [320], Hadwiger [270], Hadwiger and coauthors [282], and Valentine [683].

A set of points K in the plane is called *convex* if for each pair of points $A \in K$, $B \in K$ it is true that $AB \subset K$, where AB is the line segment joining A and B. For convenience we shall assume throughout that the convex sets are bounded and closed.

A curve with end points P, Q is called convex if its point set, together with the segment PQ, bounds a convex set. If the convex set K is bounded and has interior points, then the boundary of K is called a *closed convex curve.* Throughout, we will denote by ∂K the boundary of the set K. If all the points of K belong to ∂K, then K is a line segment.

We can prove that (a) All convex curves are piecewise differentiable (i.e., they are the union of a countable set of arcs with continuously turning tangent);

in other words, convex curves have at most a countable set of corners; (b) All bounded convex curves are rectifiable. The length of the boundary ∂K of a convex set K is called the *perimeter* of K.

2. Envelope of a Family of Lines

The envelope of a family of curves $F(x, y, \lambda) = 0$, depending on a parameter λ, is defined as that curve every point of which is a point of contact with a curve of the family. As is well known, the equation of the envelope can be obtained by eliminating the parameter λ from the two equations $F = 0$, $\partial F/\partial\lambda = 0$. We will apply this result to the case of a family of lines.

A line on the plane may be determined by its distance p from the origin and the angle ϕ of the normal with the x axis (Fig. 1.1). The equation of the line is then

$$x \cos\phi + y \sin\phi - p = 0. \tag{1.1}$$

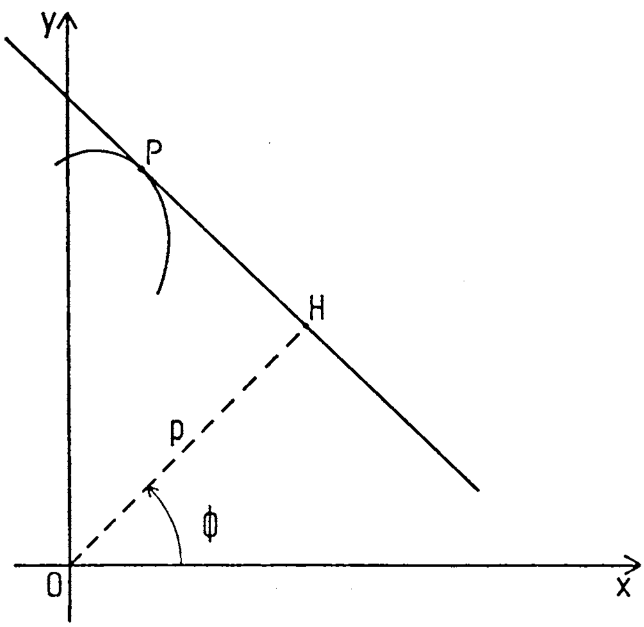

Figure 1.1.

If p is a function $p = p(\phi)$, then (1.1) is the equation of a family of lines and if we assume that $p(\phi)$ is differentiable, the envelope of the family is obtained from (1.1) and the derivative

$$-x\sin\phi + y\cos\phi - p' = 0 \qquad (p' = dp/d\phi). \tag{1.2}$$

From (1.1) and (1.2) we get the parametric representation of the envelope of the lines (1.1),

$$x = p\cos\phi - p'\sin\phi, \qquad y = p\sin\phi + p'\cos\phi. \tag{1.3}$$

These formulas give the x, y coordinates of the contact point P of the line with the envelope (Fig. 1.1). Since the coordinates of the point H in which the perpendicular through O intersects the line are $p\cos\phi$, $p\sin\phi$, it follows that

$$HP = p'. \tag{1.4}$$

Assuming that the function p is of class C^2 (recall that class C^n means n times continuously differentiable), from (1.3), it follows that $dx = -(p + p'')\sin\phi\, d\phi$, $dy = (p + p'')\cos\phi\, d\phi$. Hence $ds = |p + p''|\, d\phi$ and the radius of curvature of the envelope becomes $\rho = ds/d\phi = |p + p''|$.

If the envelope is the boundary ∂K of a convex set K and O is an interior point of K, then $p = p(\phi)$ is called the *support function* of K or the support function of the convex curve ∂K with reference to the origin O. The lines (1.1) are then the *support lines* of K. In this case we can prove that $p + p'' > 0$ (see, e.g., [63, p. 18]), and the formulas in the preceding paragraph may be written

$$ds = (p + p'')\, d\phi, \qquad \rho = p + p''. \tag{1.5}$$

It can be proved that a necessary and sufficient condition that the periodic function p be the support function of a convex set K is that $p + p'' > 0$.

From the first equation in (1.5) it follows that the length of a closed convex curve that has support function p of class C^2 is given by

$$L = \int_0^{2\pi} p\, d\phi. \tag{1.6}$$

The assumption that p is of class C^2 can be removed. We can show that formula (1.6) holds for any closed convex curve (see, e.g., [683, p. 161]).

The area of the convex set K can be also evaluated in terms of the support function. Indeed, if we consider K decomposed into elementary triangles of height p and base ds, with the point O as common vertex, we have

$$F = \tfrac{1}{2}\int_{\partial K} p\, ds = \tfrac{1}{2}\int_0^{2\pi} p(p + p'')\, d\phi \tag{1.7}$$

and by integration by parts

$$F = \frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\, d\phi. \tag{1.8}$$

3. Mixed Areas of Minkowski

Let K_1, K_2 be two bounded convex sets on the plane whose support functions with reference to O_1 and O_2 are, respectively, p_1 and p_2, assumed of class C^2. Consider the function $p(\phi) = p_1(\phi) + p_2(\phi)$. The envelope of the lines $x \cos \phi + y \sin \phi - p = 0$ is a closed curve whose radius of curvature is $\rho = p + p'' = (p_1 + p_1'') + (p_2 + p_2'') = \rho_1 + \rho_2$. Since ∂K_1 and ∂K_2 are convex curves, we have $\rho_1 > 0$, $\rho_2 > 0$ and hence $\rho > 0$. Therefore the envelope above is the boundary of a convex set K_{12}. If p_1 and p_2 are not of class C^2 the proof fails, but the result is still true: the function $p = p_1 + p_2$ is always the support function of a convex set K_{12} called the *mixed convex set* of K_1 and K_2 [63, p. 29].

The area of K_{12} has the form

$$F = \frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\, d\phi = F_1 + F_2 + 2F_{12} \tag{1.9}$$

where F_1, F_2 are the areas of K_1, K_2 and

$$F_{12} = F_{21} = \frac{1}{2}\int_0^{2\pi} (p_1 p_2 - p_1' p_2')\, d\phi \tag{1.10}$$

is the so-called *mixed area of Minkowski* of K_1 and K_2.

Integration by parts, using (1.5), gives

$$F_{12} = \frac{1}{2}\int_0^{2\pi} p_1(p_2 + p_2'')\, d\phi = \frac{1}{2}\int_{\partial K_2} p_1\, ds_2 \tag{1.11}$$

and similarly we have

$$F_{21} = \frac{1}{2}\int_{\partial K_1} p_2\, ds_1 \tag{1.12}$$

where ds_1, ds_2 are the arc elements of ∂K_1, ∂K_2 at the contact points of the support lines normal to the direction ϕ.

Note that the mixed area F_{12} does not depend on the origins O_1, O_2. In fact, if O_1 is replaced by $O_1{}^*$ such that $O_1O_1{}^* = a$ and α is the angle of $O_1O_1{}^*$ with the x axis (Fig. 1.2), the support function relative to $O_1{}^*$ is $p_1{}^* = p_1 - a\cos(\phi - \alpha)$, and using (1.10) and integrating by parts we verify that $F_{12}^* = F_{12}$. The same is clearly true if we change O_2.

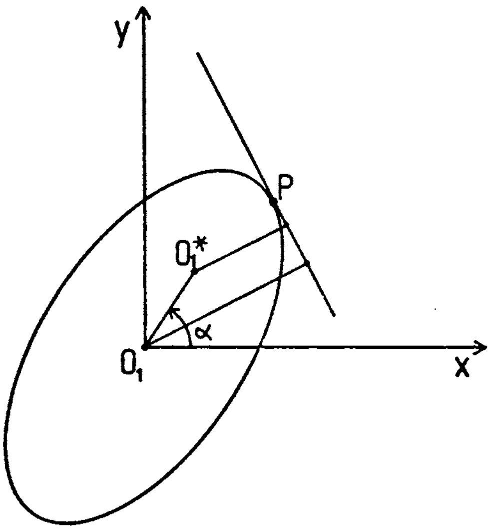

Figure 1.2.

Furthermore, the mixed area F_{12} does not change by translations of K_1 and K_2 since p_1 and p_2 remain unchanged. Assume now that one of the sets, say K_1, is rotated an angle θ about a point O. Let $O_1{}^*$ be the image of O_1. The rotation is equivalent to the rotation of angle θ about O_1 composed by the translation defined by the vector $O_1O_1{}^*$. Thus, because of the invariance of F_{12} by translations, it remains for us to consider rotations about O_1. By the rotation of angle θ about O_1 the new support function of K_1 is $p_1{}^*(\phi) = p_1(\phi - \theta)$ and the mixed area of K_2 and the rotated set $K_1{}^*$ becomes

$$F_{12}(\theta) = \frac{1}{2}\int_{\partial K_2} p_1(\phi - \theta)\, ds_2.$$

Integrating with regard to θ and using (1.6) we get

$$\int_0^{2\pi} F_{12}(\theta)\, d\theta = \frac{1}{2}L_1L_2. \qquad (1.13)$$

This formula will be useful later.

Examples 1. If K_1 is a translate of K_2, we may assume $p_1 = p_2$, $ds_1 = ds_2$, and (1.12) gives $F_{12} = F_1 = F_2$. That is, the mixed area of convex sets that are translation congruent is equal to the common area of the sets.

2. If K_2 is a circle of radius r, we have $ds_2 = r\,d\phi$ and (1.11) gives

$$F_{12} = \tfrac{1}{2}r \int_0^{2\pi} p_1\,d\phi = \tfrac{1}{2}rL_1. \tag{1.14}$$

3. Let K_1, K_2 be two line segments of lengths $2a$, $2b$, respectively, and let α be the angle between the lines which contain the segments. Using (1.10) we get $F_{12} = 2ab|\sin\alpha|$. Integrating with respect to α over the range $0, 2\pi$, we get $8ab$, in accordance with (1.13). Note that a line segment is a convex set whose perimeter is twice the length of the segment.

4. Some Special Convex Sets

A line h is said to be a *support line* of the convex set K at a point $P \in \partial K$ if $P \in h$ and K is contained in the closure of one of the two open half planes into which h cuts the plane. If ∂K possesses a tangent at P, then the support line at P coincides with the tangent. Every point on the boundary of a convex set lies on a support line and there are exactly two support lines perpendicular to a given direction.

The *breadth* $\Delta(\phi)$ of K in the direction ϕ is the distance between the two parallel support lines to K that are perpendicular to the direction ϕ and that contain K between them. If $p(\phi)$ is the support function of K, we have $\Delta(\phi) = p(\phi) + p(\phi + \pi)$ and according to (1.6) we have

$$L = \int_0^{\pi} \Delta(\phi)\,d\phi. \tag{1.15}$$

Therefore, the mean value or expected value of Δ is

$$E(\Delta) = L/\pi \tag{1.16}$$

where L is the perimeter of K. Note that the breadth $\Delta(\phi)$ may be defined as the length of the orthogonal projection of K on a line parallel to the direction ϕ. Consequently, (1.16) gives: any closed convex curve ∂K of length L can be projected on a line in such a way that the length of the projection is $\geqslant L/\pi$. It can also be projected so that such a length is $\leqslant L/\pi$.

The least of the breadths of a convex set K is called the *width* of K. We shall represent it by E. The *diameter* of K, represented by D, is the greatest distance between two points of K. It can also be defined as the greatest

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breadth of K. Since we obviously have $E \leqslant \Delta \leqslant D$, from (1.15) it follows that

$$\pi E \leqslant L \leqslant \pi D. \tag{1.17}$$

We now want to define some special convex sets that have particular interest for our purposes.

Parallel convex sets. The parallel set K_r in the distance r of a convex set K is the union of all closed circular disks of radius r the centers of which are points of K. The boundary ∂K_r is called the outer parallel curve of ∂K in the distance r. Figures 1.3, 1.4, and 1.5 show parallel sets of a segment, a triangle, and an ellipse, respectively.

Figure 1.3.

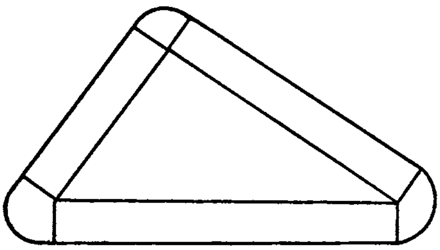

Figure 1.4.

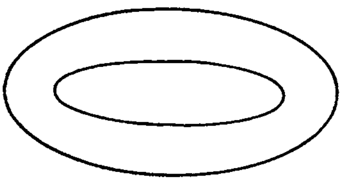

Figure 1.5.

If $p(\phi)$ is the support function of K relative to O, the support function of K_r relative to the same point O is $p(\phi) + r$, and using (1.6) and (1.8) we have that the perimeter and the area of K_r are, respectively,

$$L_r = L + 2\pi r, \qquad F_r = F + Lr + \pi r^2. \tag{1.18}$$

If ∂K is of class C^2, the radius of curvature of ∂K_r, by (1.5), is

$$\rho_r = \rho + r. \tag{1.19}$$

For values of r such that $r \leqslant \min \rho$ we can define the *interior parallel set* K_{-r} as the set whose support function is $p(\phi) - r$. Then the length of ∂K_{-r} and the area of K_{-r} are given by the same formulas (1.18) after the substitution $r \to -r$.

Sets of constant breadth. If $\Delta(\phi) = \Delta =$ constant for all ϕ, the convex set K is said to be of constant breadth. In this case we have $E = \Delta = D$ and by (1.15) the perimeter of K is given by the simple formula

$$L = \pi \Delta. \tag{1.20}$$

Furthermore, if ∂K is of class C^2, using (1.5) and the fact that $\Delta = p(\phi) + p(\phi + \pi)$, we have

$$\rho(\phi) + \rho(\phi + \pi) = \Delta. \tag{1.21}$$

Besides the circle, the simpler convex sets of constant breadth are the so-called Reuleaux polygons. Given a linear regular polygon of $2n + 1$ sides ($n = 1, 2, \ldots$), the corresponding Reuleaux polygon is formed by the circular arcs that are subtended by the sides and whose centers are the opposite vertices of the polygon. Figure 1.6 shows the Reuleaux triangle and Reuleaux pentagon. Note that each set parallel to a convex set of constant breadth is also of constant breadth.

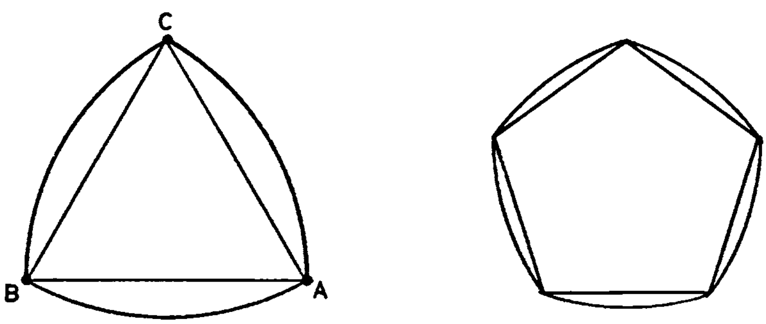

Figure 1.6.

It can be shown that any set of diameter $D \leqslant \Delta$ is a subset of a convex set of constant breadth Δ [63, p. 130].

Triangular convex sets. Convex sets of constant breadth can rotate in the interior of a square (i.e., all circumscribed rectangles are congruent squares; Fig. 1.7). There are also convex sets that can rotate inside a fixed equilateral triangle, which is equivalent to saying that all the equilateral triangles that are circumscribed to the set are congruent triangles. These sets are called *triangular sets.* For instance, the shaded spindle in Fig. 1.8 is a triangular set. Each parallel set of a triangular set is also a triangular set.

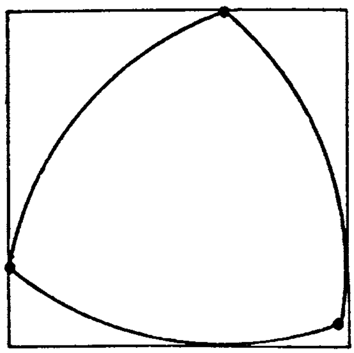

Figure 1.7.

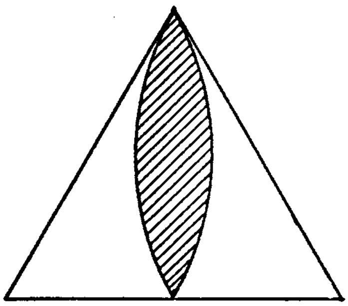

Figure 1.8.

Since the sum of the distances from an interior point of an equilateral triangle to the sides is equal to the height h of the triangle, it follows that the support function of a triangular set satisfies the condition $p(\phi) + p(\phi + 2\pi/3) + p(\phi + 4\pi/3) = h$. From this equality and (1.6) it follows that the perimeter of the triangular sets inscribed in an equilateral triangle of height h is $L = (2\pi/3)h$.

5. Surface Area of the Unit Sphere and Volume of the Unit Ball

Throughout this text we shall denote by O_n the surface area of the n-dimensional unit sphere and by κ_n the volume of the n-dimensional unit ball or solid sphere. Their values are

$$O_n = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}, \qquad \kappa_n = \frac{O_{n-1}}{n} = \frac{2\pi^{n/2}}{n\Gamma(n/2)} \tag{1.22}$$

where Γ denotes the gamma function, which satisfies the relations

$$\Gamma(n+1) = n\Gamma(n), \qquad \Gamma(n) = (n-1)! \quad (n \text{ integer}), \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \tag{1.23}$$

For instance,

$$O_0 = 2, \quad O_1 = 2\pi, \quad O_2 = 4\pi, \quad O_3 = 2\pi^2, \quad \kappa_1 = 2, \quad \kappa_2 = \pi, \quad \kappa_3 = \tfrac{4}{3}\pi.$$

6. Notes and Exercises

1. *Minimum problems concerning convex sets.* The area F, perimeter L, diameter D, and width E of a convex set are related by certain inequalities, some of which are the following.

$$L^2 \geqslant 4\pi F, \qquad \sqrt{3}\, F \geqslant E^2, \qquad L \geqslant \pi E, \qquad 2F \geqslant ED,$$

$$L \geqslant 2(D^2 - E^2)^{1/2} + 2E \arcsin(E/D),$$

$$L \leqslant 2(D^2 - E^2)^{1/2} + 2D \arcsin(E/D),$$

$$2F \leqslant E(D^2 - E^2)^{1/2} + D^2 \arcsin(E/D), \qquad 4F \leqslant 2EL - \pi E^2.$$

A complete set of inequalities remains unknown. The determination in each case of the set that satisfies the equality sign is in general a difficult problem. For references and a systematic exposition, see the articles by Kubota and Hemmi [349], Ohmann [462], Santaló [576], and Sholander [606]. Several results on convex sets can be found in the book *Convexity* (V. Klee, ed.; Amer. Math. Soc., Providence, R.I., 1963).

2. *Polar reciprocal convex sets.* Let K be a bounded convex set in the plane. Let O be an interior point of K and let P be the boundary point of K in the direction ϕ from O. The function $h^*(\phi) = |OP|^{-1}$ is the support function of a convex set K^* called the polar reciprocal of K with respect to O. The polar reciprocal convex sets have several applications to the geometry of numbers (see the work of C. G. Lekkerkerker [361] and Cassels [92]). We state the following properties:

(a) The mixed area $F_{12}(K, K^*)$ satisfies $F_{12}(K, K^*) \geqslant \pi$ with equality if and only if K is a circle of center O [192].

(b) If K has O as center of symmetry, then the areas $F(K)$ and $F(K^*)$ satisfy the inequalities

$$\tfrac{1}{2}\pi^2 \leqslant F(K)\cdot F(K^*) \leqslant \pi^2. \tag{1.24}$$

The affine invariant $F(K)\cdot F(K^*)$ has been studied for closed curves not necessarily convex, for which the polar angle is a monotone function of the arc length, and applied to inequalities related to periodic solutions of differential equations by Guggenheimer [255]. For convex curves in the plane, see [296] and for applications to differential equations see [445–448].

The inequalities (1.24) may be extended to centrally symmetric convex sets in n-dimensional euclidean space E_n. Calling $V(K)$ and $V(K^*)$ the respective volumes, we have

$$\kappa_n{}^2 n^{-n/2} \leqslant V(K)\cdot V(K^*) \leqslant \kappa_n{}^2 \tag{1.25}$$

where κ_n is the volume of the unit ball in E_n (1.22). Inequalities (1.25) have their origin in the work of Mahler [387] and were improved by Bambah [19] and Santaló [556]. The right-hand inequality in (1.25) is the best possible for all convex bodies, not necessarily centrally symmetric, and equality holds for the n-dimensional ellipsoid centered at O. For centrally symmetric convex bodies, the left-hand equality in (1.25) holds if and only if K is a parallelotope