1

SETS AND FUNCTIONS

1.1. Sets and numbers

It has been recognized since the latter part of the nineteenth century that the idea of number (real and complex), and therefore all analysis, is based on the theory of sets. In modern analysis the dependence is explicit, for the language and algebra of sets are in constant use.

The reader is likely to be familiar with the intuitive notion of a set and with the basic operations on sets. In this section we therefore confine ourselves to fixing the terminology and recapitulating the results that will be used subsequently. The notes at the end of the chapter refer to books which develop set theory systematically from explicitly stated axioms.

There are synonyms for the word set such as *collection* and *space*. The members of a set are also called its *elements* or *points*. The statement that a belongs to (or is a member of) the set A is written

$$a \in A.$$

If a does not belong to A we write

$$a \notin A.$$

We denote by $\varnothing$ *the empty set*, namely the set which has no members.

Inclusion of sets. Suppose that every member of the set A also belongs to the set B, i.e. that

$$x \in A \Rightarrow x \in B. \tag{1.11}$$

Then we say that A is contained in B and write

$$A \subset B \quad \text{or} \quad B \supset A;$$

the set A is said to be a *subset* of B. Note that any set is a subset of itself. Also the empty set is a subset of every set; for (1.11) is logically equivalent to

$$x \notin B \Rightarrow x \notin A$$

and, when A is the empty set, this implication clearly holds for any set B.

If $A \subset B$ and $B \subset A$, i.e.

$$x \in A \Leftrightarrow x \in B,$$

then A and B have the same members and we write

$$A = B.$$

If $A \subset B$ and $A \neq B$, we can call A a *proper* subset of B.

Union and intersection. Given any two sets A, B, the set which consists of all the elements belonging to A or to B (or to both) is called the *union* of A and B and is denoted by

$$A \cup B.$$

The set consisting of the elements which belong to both A and B is called the *intersection* of A and B and is denoted by

$$A \cap B.$$

The sets A, B are said to be *disjoint* if they have no elements in common, i.e. if $A \cap B = \varnothing$.

Theorem 1.11. *The following identities hold for any sets A, B, C.*

(i) $A \cup B = B \cup A, \quad A \cap B = B \cap A;$

(ii) $A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C;$

(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$

Proof. We illustrate the argument used for establishing these identities by proving the first result in (iii).

Let x be any element of $A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so that $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B$ and $x \in C$ so that again $x \in (A \cup B) \cap (A \cup C)$. We have therefore shown that

$$A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C). \tag{1.12}$$

Now let $y \in (A \cup B) \cap (A \cup C)$. Then $y \in A \cup B$ and $y \in A \cup C$. It follows that $y \in A$ or that $y \in B$ and $y \in C$, i.e. that $y \in A$ or $y \in B \cap C$. Hence $y \in A \cup (B \cap C)$ and so

$$(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C). \tag{1.13}$$

The inclusion relations (1.12) and (1.13) now yield the required identity. ∎

The laws of operation with $\cup$ and $\cap$ on sets have some likeness to those with $+$ and $\times$ on numbers. In fact $\cup$ and $\cap$ have replaced the signs for sum and product formerly used in the algebra of sets. The likeness is only partial: the identity $A \cup A = A$ has no analogue in numbers; the second law in theorem 1.11 (iii) has an analogue, but not the first.

The definitions of union and intersection previously given may be extended. If $\mathscr{C}$ is an arbitrary collection of sets, the union of these sets, denoted by

$$\bigcup_{A \in \mathscr{C}} A,$$

is the set consisting of all those elements which belong to at least one of the sets A. The intersection of the sets of $\mathscr{C}$, denoted by

$$\bigcap_{A \in \mathscr{C}} A,$$

is the set consisting of the elements which belong to all the sets A.

We require one more operation with sets. If A, B are any two sets, the *difference*

$$A - B$$

is the set consisting of those elements which belong to A but not to B. Note that the definition does not require B to be a subset of A. However a particularly important case occurs when all sets under consideration are subsets of a given set X. Then the set $X - A$ is called the *complement* of A (relative to X) and we shall denote it by A'. Clearly $(A')' = A$.

Theorem 1.12. *For any collection $\mathscr{C}$ of subsets of a set X,*

$$\Big(\bigcup_{A \in \mathscr{C}} A\Big)' = \bigcap_{A \in \mathscr{C}} A' \quad \text{and} \quad \Big(\bigcap_{A \in \mathscr{C}} A\Big)' = \bigcup_{A \in \mathscr{C}} A'.$$

The proof is left to the reader.

Theorem 1.12 shows that the operations $\bigcup$ and $\bigcap$ are complementary and that an algebraic identity involving $\bigcup$ and $\bigcap$ will admit of a *dual* identity obtained by interchanging $\bigcup$, $\bigcap$. For example, in each pair of identities in theorem 1.11 the second follows from the first.

We shall take for granted that the reader is familiar with the systems of real and complex numbers. There are well known and simple methods for obtaining the complex numbers from the real numbers. For our purposes it is only necessary to postulate the existence of the system of real numbers as one satisfying certain axioms which are

given in C1 (11–12) and are restated in the notes at the end of the chapter. However, it is possible to construct the real numbers from much more primitive objects. Again the notes contain some remarks on the subject.

The set of real numbers will be denoted by R^1, the set of complex numbers by Z. Our notation for the *open interval* in R^1 which consists of the points x such that $a < x < b$ is (a, b). The *closed interval* of points x such that $a \leqslant x \leqslant b$ is written $[a, b]$. The intervals defined by $a \leqslant x < b$ and $a < x \leqslant b$ are written $[a, b)$ and $(a, b]$. The infinite interval $[a, \infty)$ consists of the points x such that $x \geqslant a$. Corresponding interpretations are put on the expressions (a, ∞), $(-\infty, a)$, $(-\infty, a]$. Finally, $(-\infty, \infty)$ is R^1.

Exercises 1(*a*)

1. Prove that

(i) $A \cup B = A \Leftrightarrow A \supset B$; (ii) $A \cap B = A \Leftrightarrow A \subset B$.

2. Prove that

$$(B \cup C) \cap (C \cup A) \cap (A \cup B) = (B \cap C) \cup (C \cap A) \cup (A \cap B).$$

3. Show that $A - B = A \cap B'$. Hence or otherwise prove that

(i) $(A-B) \cap (A-C) = A-(B \cup C) = (A-B)-C$;

(ii) $(A-B) \cup (A-C) = A-(B \cap C)$;

(iii) $(A-C) \cap (B-C) = (A \cap B)-C$;

(iv) $(A-C) \cup (B-C) = (A \cup B)-C$.

4. Prove that

(i) $(A-B) \cap C = (A \cap C)-(B \cap C)$;

(ii) $(A-B) \cup C = (A \cup C)-(B \cup C) \Leftrightarrow C = \varnothing$.

5. Prove that

(i) $(A \cup B)-B = A \Leftrightarrow A \cap B = \varnothing$;

(ii) $(A-B) \cup B = A \Leftrightarrow A \supset B$.

6. The *symmetric difference* $A \triangle B$ of the sets A, B is defined as the set of elements in A or in B, but not in both. Show that

$$A \triangle B = (A-B) \cup (B-A) = (A \cup B)-(A \cap B) = (A \cup B) \cap (A' \cup B').$$

Deduce that

(i) $A \triangle B = \varnothing \Leftrightarrow A = B$; (ii) $A' \triangle B' = A \triangle B$.

7. Prove that

(i) $A \triangle B = B \triangle A$; (ii) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

8. Prove that

(i) $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$;

(ii) $(A \triangle B) \cup C = (A \cup C) \triangle (B \cup C) \Leftrightarrow C = \varnothing$;

(iii) $(A \triangle B) \cup (A \triangle C) = (A \cup B \cup C) \cap (A' \cup B' \cup C')$.

9. Show that $A \triangle B = C \triangle D \Leftrightarrow A \triangle C = B \triangle D$.

10. When E is a finite set (i.e. one with finitely many elements), denote by $|E|$ the number of its elements.

Show that, if the sets A, B are finite, then

$$|A| + |B| = |A \cup B| + |A \cap B|.$$

1.2. Ordered pairs and Cartesian products

A set with elements $a, b, c, \ldots$ is often denoted by the symbol

$$\{a, b, c, \ldots\}. \tag{1.21}$$

The notation calls for a number of comments.

(i) In such a listing of elements it is immaterial whether a particular element appears once or several times. This convention is purely a matter of convenience. For instance it allows us to denote the set of roots of a complex quadratic equation by $\{a, b\}$, whether $a \neq b$ or $a = b$.

(ii) The order in which $a, b, c, \ldots$ are written in $\{a, b, c, \ldots\}$ has no significance. For example $\{a, b\} = \{b, a\}$.

(iii) It is important to distinguish between the object a and the set $\{a\}$, i.e. the set whose only member is a. Thus the set $\{\varnothing\}$ has one element, while $\varnothing$ has none.

The notation (1.21) has a useful variant. Given a set X, denote by $P(x)$ a statement, which is either true or false, about the element x of X. The set of all $x \in X$ for which $P(x)$ is true is written

$$\{x \in X | P(x)\},$$

or simply

$$\{x | P(x)\}$$

if the identity of the set X is clear from the context.

Illustrations

(i) $\{x \in Z | x^2 + 1 = 0\} = \{i, -i\}$;

(ii) $\{x \in R^1 | x^2 + 1 = 0\} = \varnothing$;

(iii) $\{x \in R^1 | a \leqslant x \leqslant b\} = [a, b]$.

The notion of a set does not involve any ordering among the elements. For instance in the set $\{a, b\}$, a and b have the same status.

The *ordered pair* (a, b) is the set $\{a, b\}$ together with the ordering 'first a, then b'. Thus (a, b) and (b, a) are different unless $a = b$. (The context will determine whether the expression (a, b) stands for an ordered pair or an open interval.)

The intuitive concept of an ordered pair (a, b) may be formalized by the definition

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

(See exercise 1(*b*), 1.)

An ordered set $(a_1, \ldots, a_n)$ of any finite number n (> 1) of elements is defined in a similar way.

Definition. *Given the non-empty sets $X_1, \ldots, X_n$, their Cartesian product*

$$X_1 \times \ldots \times X_n \tag{1.22}$$

is the collection of all ordered sets $(x_1, \ldots, x_n)$ such that

$$x_1 \in X_1, \ldots, x_n \in X_n.$$

If $X_1 = \ldots = X_n = X$, the set (1.22) *is denoted by X^n; X^1 is taken to be X.*

For $n > 1$, $R^n = R^1 \times \ldots \times R^1$ (n factors) is the set of all ordered sets $(x_1, \ldots, x_n)$ of real numbers. We define an *interval* in R^n as a set

$$I_1 \times \ldots \times I_n,$$

where $I_1, \ldots, I_n$ are intervals in R^1. For example $[a, b] \times [c, d]$ is the set of points $(x, y) \in R^2$ such that $a \leqslant x \leqslant b$ and $c \leqslant y \leqslant d$.

1.3. Functions

Let X, Y be two non-empty sets. A *relation from X to Y* is a subset of $X \times Y$. If a relation f, i.e. a subset of $X \times Y$, is such that, *for every $x \in X$, there is one and only one member (x, y) of f*, then f is said to be a *function on X to* (or *into*) Y; we express this symbolically by writing

$$f: X \to Y.$$

The set X is called the *domain* of f.

In some contexts the terms *mapping*, *transformation*, or *operator* are often used as synonyms for function. Let (x, y) be an element of the function $f: X \to Y$. Then we say that y is the *value* of f at x (or the *image* of x under f) and we write $y = f(x)$. If E is any subset of X, the subset of Y

$$\{y \in Y | y = f(x) \quad \text{and} \quad x \in E\}$$

is called the image of E under f; it is denoted by $f(E)$. The set $f(X)$ is called the *range* of f. Note that $f(X)$ may be a proper subset of Y.

If a function f has its domain in X and its range in Y, we say that f is a function *from* X to Y. A function from R^1 to R^1 will be called a *real function*, a function from Z to Z a *complex function*.

Illustrations

(i) We obtain a function on X to Y whenever we have a rule for assigning a single value y in Y to each x of X. Thus the equation $x^2+y^3 = 1$ defines a function on R^1 to R^1 whose range is $(-\infty, 1]$.

(ii) The equation $x^2+y^2 = 1$ defines a relation but not a function on $[-1, 1]$ to $[-1, 1]$ because, whenever $-1 < x < 1$, it is satisfied by two values of y.

(iii) The logarithmic function is the function on $(0, \infty)$ to R^1 which assigns the value $\log x$ to the positive number x.

(iv) The sequence $a_1, a_2, a_3, \ldots$ is a function whose domain is the set of positive integers exhibited as suffixes.

More generally, given two arbitrary non-empty sets A, I and a function on I to A, we sometimes find it convenient to write the image of a member i of I in the form a_i.

(v) Denote by Φ the set of all real functions ϕ defined by an equation of the form

$$\phi(x) = \alpha x+\beta \quad (x \in R^1),$$

where α, β are real numbers. A function $f\colon R^2 \to \Phi$ is obtained by assigning to each point $(a, b) \in R^2$ the function $\phi_{a,b}$ given by $\phi_{a,b}(x) = ax+b \quad (x \in R^1)$.

Let f be a function on X into Y. If $f(X) = Y$, i.e. if every $y \in Y$ is the image under f of at least one $x \in X$, then we say that f is *surjective*, or that f maps X *onto* Y.

If different elements of X have different images, i.e. if $f(x) = f(x')$ implies $x = x'$, then f is called *injective*.

If f is both surjective and injective, so that every value y in Y is assumed once and once only, then f is said to be *bijective*. A bijective function is also called a *bijection*; it effects a *one-to-one correspondence* between its domain and range.

Consider again the illustrations above.

(i) The function maps R^1 onto $(-\infty, 1]$; it is not injective.

(iii) The logarithmic function is a bijection on $(0, \infty)$ to R^1.

(v) This function is a bijection on R^2 to Φ.

Inverse functions. Suppose that the function f has domain X and range Y. To f, which is a subset of $X \times Y$, there corresponds the subset of $Y \times X$ defined as

$$\{(y, x) | (x, y) \in f\}. \tag{1.31}$$

Plainly this is a function with domain Y and range X if and only if f is bijective. When f is bijective, (1.31) is called the *inverse function* of f and is denoted by f^{-1}. For instance the logarithmic function has

an inverse with domain $(-\infty, \infty)$ and range $(0, \infty)$; the inverse is the exponential function.

If the function $f: X \to Y$ is injective, then $f: X \to f(X)$ is bijective and $f^{-1}: f(X) \to X$ is a function *from* Y to X. For example the function $f: R^1 \to R^1$ defined by

$$f(x) = \frac{1}{1+e^x}$$

has the inverse $f^{-1}: (0, 1) \to R^1$ given by

$$f^{-1}(y) = \log\left(\frac{1}{y} - 1\right).$$

It is clear that, if f has an inverse, then f^{-1} has an inverse and $(f^{-1})^{-1} = f$.

Inverse images. Let $f: X \to Y$ be an arbitrary function and let E be any subset of Y. We denote by $f^{-1}(E)$ the set

$$\{x \in X | f(x) \in E\}.$$

This set is called the *inverse image* of E. It is important to note that the definition of $f^{-1}(E)$ does not presuppose the existence of the inverse function f^{-1}. However, when f^{-1} does exist, then $f^{-1}(E)$ is the image of E under f^{-1}.

If f is the function in (i) on p. 7 then, for example,

$$f^{-1}((0, 1]) = (-1, 1);$$

the *function* f^{-1} does not exist.

Composition of functions. Given the functions $f: X \to Y$ and $g: Y \to W$, the function $h: X \to W$ defined by

$$h(x) = g\{f(x)\} \quad (x \in X)$$

is called the *composition* of g and f and is written $g \circ f$.

For three functions $f_1: X_1 \to X_2$, $f_2: X_2 \to X_3$, $f_3: X_3 \to X_4$, we clearly have

$$f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1$$

so that the expression $f_3 \circ f_2 \circ f_1$ has a meaning. The process of composition may be extended to any number of functions.

For a non-empty set A, denote by i_A the *identity function* on A defined by

$$i_A(x) = x \quad (x \in A).$$

If, now, $f: X \to Y$ is any function,

$$f \circ i_X = f, \quad i_Y \circ f = f;$$

and, when f is bijective,

$$f^{-1} \circ f = i_X, \quad f \circ f^{-1} = i_Y.$$

Theorem 1.31.

(i) *If the function* $f: X \to Y$ *is surjective and if there is a function* $g: Y \to X$ *such that* $g \circ f = i_X$, *then* f *is bijective and* $g = f^{-1}$.

(ii) *If the function* $f: X \to Y$ *is bijective and if the function* $g: Y \to X$ *is such that* $f \circ g = i_Y$, *then* $g = f^{-1}$.

Proof.

(i) If $f(x) = f(x')$, then

$$x = g\{f(x)\} = g\{f(x')\} = x'$$

and so f is injective as well as surjective. We now have

$$g = g \circ i_Y = g \circ (f \circ f^{-1}) = (g \circ f) \circ f^{-1} = i_X \circ f^{-1} = f^{-1}.$$

(ii) $g = i_X \circ g = (f^{-1} \circ f) \circ g = f^{-1} \circ (f \circ g) = f^{-1} \circ i_Y = f^{-1}$. |

In theorem 1.31 (ii) the bijectiveness of f is not a consequence of surjectiveness, as it is in part (i). For instance let $X = \{a, b\}$ $(a \neq b)$, $Y = \{c\}$ and let f, g be defined by $f(a) = f(b) = c$, $g(c) = a$. Then f is surjective and $f \circ g = i_Y$, but f is not bijective.

If the functions $f: X \to Y$, $g: Y \to W$ are bijective, then $g \circ f: X \to W$ is bijective. It follows from theorem 1.31, or it may be proved directly, that

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Restriction and extension of functions. Suppose that $X_1 \supset X_2$ and that $f: X_1 \to Y$, $g: X_2 \to Y$ are functions such that $f(x) = g(x)$ for all $x \in X_2$. We then say that g is *the restriction* of f to X_2 and that f is *an extension* of g to X_1. For example let $f_1: R^1 \to R^1$ and $f_2: R^1 \to R^1$ be given by

$$f_1(x) = x, \quad f_2(x) = |x| \quad (x \in R^1).$$

The function $g: [0, \infty) \to R^1$ defined by

$$g(x) = x \quad (x \geqslant 0)$$

is the restriction to $[0, \infty)$ of both f_1 and f_2; and f_1, f_2 are both extensions of g to R^1.

Exercises 1(*b*)

1. Show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$.

2. Prove that

(i) $(A \cup B) \times C = (A \times C) \cup (B \times C)$,

(ii) $(A \cap B) \times C = (A \times C) \cap (B \times C)$,

(iii) $(A - B) \times C = (A \times C) - (B \times C)$.

3. Prove that $A \times B = B \times A$ if and only if $A = B$. Is $(A \times A) \times A$ necessarily the same set as $A \times (A \times A)$?

4. Show that $\{a\} \times \{a\} = \{\{\{a\}\}\}$.

5. Let $\mathscr{C}$ be a collection of subsets of a set X. Prove that, for any function $f: X \to Y$,

$$\text{(i) } f(\bigcup_{A \in \mathscr{C}} A) = \bigcup_{A \in \mathscr{C}} f(A), \quad \text{(ii) } f(\bigcap_{A \in \mathscr{C}} A) \subset \bigcap_{A \in \mathscr{C}} f(A).$$

Show that, if f is injective, then identity holds in (ii); and that f is injective if

$$f(A \cap B) = f(A) \cap f(B)$$

for all subsets A, B of X.

6. Show that, if $f: X \to Y$ is any function, then

$$f(A - B) \supset f(A) - f(B)$$

for all subsets A, B of X; and that identity holds for all A, B if and only if f is injective.

7. Let $\mathscr{C}$ be a collection of subsets of a set Y. Show that, for any function $f: X \to Y$,

$$\text{(i) } f^{-1}(\bigcup_{B \in \mathscr{C}} B) = \bigcup_{B \in \mathscr{C}} f^{-1}(B), \quad \text{(ii) } f^{-1}(\bigcap_{B \in \mathscr{C}} B) = \bigcap_{B \in \mathscr{C}} f^{-1}(B).$$

8. Show that, if $f: X \to Y$ is any function, then

$$f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$$

for all subsets C, D of Y.

9. Show that, if $f: X \to Y$ is any function, then

$$\text{(i) } f^{-1}(f(A)) \supset A, \quad \text{(ii) } f(f^{-1}(B)) \subset B$$

for all subsets A of X and all subsets B of Y. Prove also that in (i) identity holds for all A if and only if f is injective; and in (ii) identity holds for all B if and only if f is surjective.

10. The functions $f_1: X_1 \to X_2$, $f_2: X_2 \to X_3$, $f_3: X_3 \to X_4$ are such that the compositions $f_2 \circ f_1: X_1 \to X_3$ and $f_3 \circ f_2: X_2 \to X_4$ are both bijective. Show that f_1, f_2, f_3 are all bijective.

11. Let R be a relation from X to X (i.e. a subset of $X \times X$) and write xRy when $(x, y) \in R$. The relation is said to be

(i) *reflexive* if $x \in X \Rightarrow xRx$;
(ii) *symmetric* if $xRy \Rightarrow yRx$;
(iii) *transitive* if xRy and $yRz \Rightarrow xRz$.

Criticize the following 'argument'. The relation R is known to be symmetric and transitive. Writing x for z in (iii) and using (ii) we obtain $xRy \Rightarrow xRx$. Hence R is reflexive.

1.4. Similarity of sets

If there is a one-to-one correspondence between two sets A, B, i.e. if there exists a bijection $f: A \to B$, then the two sets are said to be *similar* and we write

$$A \sim B.$$