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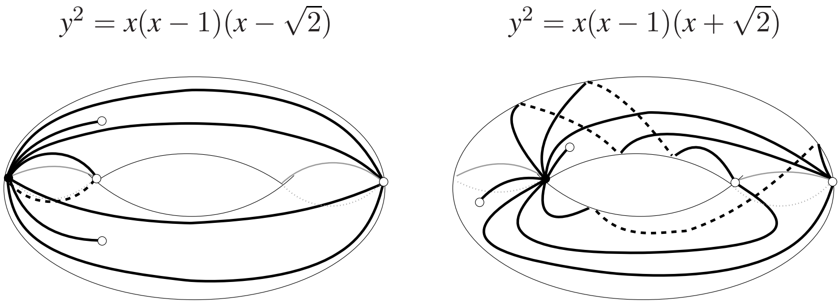
$y^2 = x(x-1)(x-\sqrt{2})$
$y^2 = x(x-1)(x+\sqrt{2})$

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Introduction to Compact Riemann Surfaces and Dessins d'Enfants

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For our families

Emi and Álvaro

and

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Ernesto Girondo and Gabino González-Diez
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Preface

The present text is an expanded version of the lecture notes for a course on Riemann surfaces and dessins d'enfants which the authors have taught for several years to students of the masters degree in mathematics at the Universidad Autónoma de Madrid.

Riemann surfaces are an ideal meeting ground for several branches of mathematics. For example, a student taking a course like this will encounter concepts of algebraic topology (fundamental group, theory of covering spaces, monodromy), elements of Riemannian geometry (geodesics, isometries, tessellations), objects belonging to algebra and algebraic geometry (field extensions, algebraic curves, valuations), definitions belonging to arithmetic geometry (fields of moduli and definition of an algebraic variety), some elementary graph theory (dessins d'enfants), tools of (complex) analysis (Weierstrass functions and Poincaré series) and some of the most relevant groups in analytic number theory (principal congruence subgroups).

One of the main features of the theory of Riemann surfaces is that there is a bijective correspondence between isomorphism classes of compact Riemann surfaces and isomorphism classes of complex algebraic curves. Establishing this correspondence requires proving first that a Riemann surface has enough meromorphic functions to separate its points. This can be done by either applying the Riemann–Roch Theorem or using the Uniformization Theorem to construct these functions by means of Poincaré series (Weierstrass functions, in the genus one case). In this book we have chosen the second option, thereby introducing Fuchsian groups, the third member of this trinity of equivalent objects. The Uniformization Theorem is the only result we assume without proof. On the one hand it has a very simple statement, and on the other the students at this level have become familiar with the result for open sets of the complex plane in their first course on complex analysis, in which they probably have had to find Riemann mappings between different simply connected regions of the plane, and at the same time learnt that, due to Liouville's theorem, this is impossible if one of the regions is the whole plane. Otherwise the prerequisites have been kept to a minimum. An undergraduate student who has taken courses on vector calculus, point-set topology, field theory and complex analysis should be in a position to follow a course based in this book. Whenever we thought that an example could help to understand the theory we included one. About one third of the pages of the book are devoted to worked out examples and illustrative pictures.

The text is divided in two parts, consisting of two chapters each. In the first part we give an elementary introduction to the theory of compact Riemann surfaces. The main goal of these two first chapters is to establish the equivalence between compact Riemann surfaces, compact algebraic curves and Fuchsian groups of finite type (or lattices of $\mathbb{C}$, in the genus 1 case). We have made an effort to work out in detail the Riemann surface structure associated to a number of particularly interesting curves (hyperelliptic, Fermat, Klein, etc.) and Fuchsian groups (triangle groups, groups with special symmetric fundamental domains, etc.). When possible we have shown the link between these objects. This first part could serve by itself as a textbook for an elementary introduction to the theory of compact Riemann surfaces from the point of view of algebraic curves and Fuchsian groups. The results in it are therefore very classical as they go back to Riemann, Weierstrass, Hurwitz, Poincaré, Klein, etc.

On the contrary, the theory presented in the second part (third and fourth chapters) is much more modern. It was launched by Grothendieck in the 1980s, in his now famous *Equisse d'un programme*, following the disclosure of Belyi's celebrated theorem, whose amazingly simple proof seems to have impressed him deeply (in his own words: '*jamais sans doute un résultat profond et déroutant ne fut démontré en si peu de lignes!*'†). Here we focus on those Riemann surfaces whose corresponding algebraic equa-

† Never, without a doubt, was such a deep and disconcerting result proved in so few lines!

tion has coefficients in $\overline{\mathbb{Q}}$, the field of algebraic numbers. This is an extraordinarily beautiful theory. Without needing to invoke all the far-reaching ideas and proposals raised by Grothendieck in the Equisse, some of them beyond the scope of this book (Grothendieck–Teichmüller Theory, etc), it is impossible to resist the attraction of Grothendieck's correspondence:

$$\left\{\begin{array}{c}\text{graphs dividing an}\\ \text{orientable surface}\\ \text{into a disjoint}\\ \text{union of cells}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{algebraic curves } C \text{ endowed}\\ \text{with a function } f \text{ ramified}\\ \text{over three values, both with}\\ \text{coefficients in } \overline{\mathbb{Q}}\end{array}\right\}$$

whose proof is the second part's main goal. We stress the fact that the objects on the left-hand side, called dessins d'enfants (child's drawings) by Grothendieck, are purely topological, whereas those on the right-hand side have an arithmetic nature. It turns out that this correspondence is a consequence of Belyi's correspondence:

$$\left\{\begin{array}{c}\text{algebraic curves}\\ \text{with coefficients in } \overline{\mathbb{Q}}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Riemann surfaces with}\\ \text{a meromorphic function}\\ \text{ramified over three values}\end{array}\right\}$$

The discovery of this theory seems to have made a big impact on Grothendieck:

Cette découverte, qui techniquement se réduit à si peu de choses, a fait sur moi une impression très forte, et elle représente un tournant décisif dans le cours de mes réflexions, un déplacement notamment de mon centre d'intérêt en mathématique, qui soudain s'est trouvé fortement localisé. Je ne crois pas qu'un fait mathématique m'ait jamais autant frappé que celui-là, et ait eu un impact psychologique comparable.†

Let us observe that via the natural action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on pairs (C, f), as above, Grothendieck's correspondence implies an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins. Thus, this theory can be applied to understand the structure of the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, which one can regard as embodying the whole of classical Galois theory over $\mathbb{Q}$.

The proof of Belyi's correspondence is given in Chapter 3. In

† This discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections, a shift in particular of my centre of interest in mathematics, which suddenly found itself strongly focused. I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact (translation by Schneps and Lochak).

one direction (algebraic curves defined over $\overline{\mathbb{Q}}$ admit functions with only three branching values) the result follows from a surprisingly simple construction due to Belyi. For the opposite direction he invokes a criterion of rationality due to Weil. However, the machinery used by Weil in the proof of this criterion exceeds by far the elementary level we want to retain in this book. So, with the tools we have at our disposal, we work out a (much) weaker criterion which nevertheless satisfies our needs. Grothendieck's correspondence itself is proved in Chapter 4. In it we also study the first properties of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. A good handful of examples are explicitly described. All in all this chapter leads readers to the boundary of current research in the subject.

Those who wish to pursue this theory are adviced to consult the book by Lando and Zvonkin, the conference proceedings edited by Schneps and Lochak, the survey articles by Jones–Singerman, Shabat–Voevodsky, Wolfart, Jones, Cohen–Itzykson–Wolfart and Lochak, and, from a different point of view, the monograph by Bowers–Stephenson, which are all included in the references to this book.

As for readers wishing to study the subject of Riemann surfaces beyond or complementing the introduction given in the first part of this book, there are many excellent references available. Among them are the books by Jones–Singerman, Beardon, Farkas–Kra, Siegel, Jost, Forster, Miranda, Buser and Kirwan.

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