1

Compact Riemann surfaces and algebraic curves

1.1 Basic definitions

1.1.1 Riemann surfaces – examples

Definition 1.1 A *topological surface* X is a Hausdorff topological space provided with a collection $\{\varphi_i : U_i \longrightarrow \varphi_i(U_i)\}$ of homeomorphisms (called *charts*) from open subsets $U_i \subset X$ (called *coordinate neighbourhoods*) to open subsets $\varphi_i(U_i) \subset \mathbb{C}$ such that:

(i) the union $\bigcup_i U_i$ covers the whole space X; and
(ii) whenever $U_i \cap U_j \neq \emptyset$, the *transition function*

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

is a homeomorphism (Figure 1.1).

A collection of charts fulfilling these properties is called a (topological) *atlas*, and the inverse φ_i^{-1} of a chart is called a *parametrization*.

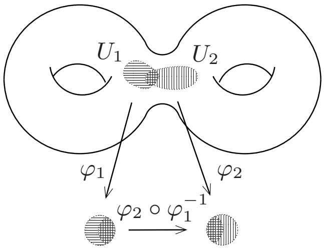

Fig. 1.1. The transition function between two coordinate charts.

Definition 1.2 A *Riemann surface* is a connected topological surface such that the transition functions of the atlas are holomorphic mappings between open subsets of the complex plane $\mathbb{C}$ (rather than mere homeomorphisms).

Example 1.3 The simplest Riemann surfaces are those defined by a single chart. Any connected open subset U in the plane is obviously a Riemann surface with the atlas consisting simply of the chart (U, Id). Particularly interesting cases are the whole *complex plane* $\mathbb{C}$, the *unit disc* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the *upper halfplane* $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$.

Example 1.4 One chart is not enough to define a Riemann surface structure on the sphere

$$\mathbb{S}^2 = \left\{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 + t^2 = 1\right\}$$

since the whole sphere is not homeomorphic to an open subset of the plane. But one can consider the following two charts:

$$U_1 = \mathbb{S}^2 \setminus \{(0,0,1)\}, \quad \varphi_1(x, y, t) = \frac{x}{1-t} + i\frac{y}{1-t}$$

$$U_2 = \mathbb{S}^2 \setminus \{(0,0,-1)\}, \quad \varphi_2(x, y, t) = \frac{x}{1+t} - i\frac{y}{1+t}$$

From the identity $\dfrac{x - iy}{1+t} = \dfrac{1-t}{x+iy}$ it follows that the transition function is $\varphi_2 \circ \varphi_1^{-1}(z) = 1/z$, where z denotes the (complex) variable in $\varphi_1(U_1)$. Note that the domain where $\varphi_2 \circ \varphi_1^{-1}$ is defined is $\varphi_1(U_1 \cap U_2) = \mathbb{C} \setminus \{0\}$.

Example 1.5 The name *Riemann sphere*, or *extended complex plane* is usually given to the Riemann surface $\widehat{\mathbb{C}}$ defined as follows. Add an aditional point to the complex plane $\mathbb{C}$, and denote it as ∞ (the notation indicates what the topology at this additional point is going to be: one gets close to ∞ by escaping from every point in the plane). A collection of fundamental neighbourhoods of ∞ is provided by the family of sets $D(\infty, R) = \{z \in \mathbb{C},\ |z| > R\} \cup \{\infty\}$.

Denote $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The Riemann surface structure is determined by the following two charts:

$$U_1 = \mathbb{C}, \quad \psi_1(z) = z$$

$$U_2 = \widehat{\mathbb{C}} \setminus \{0\}, \quad \psi_2(z) = \begin{cases} 1/z \text{ if } z \neq \infty \\ 0 \text{ if } z = \infty \end{cases}$$

Example 1.6 The complex projective line $\mathbb{P}^1 := \mathbb{P}^1(\mathbb{C})$ admits a Riemann surface structure with the following two charts:

$$U_0 = \{[z_0 : z_1] \text{ with } z_0 \neq 0\}, \ \phi_0([z_0 : z_1]) = \frac{z_1}{z_0}$$

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The next examples are described in terms of algebraic (polynomial) equations. We shall refer to them as *curves*.

Example 1.7 (Hyperelliptic curves I) Consider first the algebraic equation $y^2 = \prod_{k=1}^{2g+1}(x - a_k)$, where $\{a_k\}_{k=1}^{2g+1}$ is a collection of $2g+1$ distinct complex numbers, and let

$$\overset{\circ}{S} = \left\{ (x,y) \in \mathbb{C}^2 : y^2 = \prod_{k=1}^{2g+1}(x - a_k) \right\}$$

We shall now define a chart (U, φ) around each given point $P_0 = (x_0, y_0)$. As it is easier, we rather describe parametrizations:

- If $x_0 \neq a_i$ for all i (and so $y_0 \neq 0$), we take

$$\varphi^{-1}(z) = \left(z + x_0, \sqrt{\prod_{k=1}^{2g+1}(z + x_0 - a_k)} \right)$$

  defined in the disc $\{|z| < \varepsilon\}$ with ε small enough for z not to reach any of the values a_i. The branch of the square root is chosen so that its value at $z = x_0$ equals y_0 (and not $-y_0$).
- For $P_0 = (a_j, 0)$, we take

$$\varphi_j^{-1}(z) = \left(z^2 + a_j, z\sqrt{\textstyle\prod_{k\neq j}(z^2 + a_j - a_k)} \right), \ |z| < \varepsilon$$

  again with ε small enough to guarantee that $z^2 + a_j$ does not reach a_k for every $k \neq j$. Note that if we had taken the first coordinate to be $z + a_j$ then the second one would not be a well-defined holomorphic function in $|z| < \varepsilon$. Also note that the choice of the branch of the square root is irrelevant. If we had defined a different parametrization $\widetilde{\varphi}_j^{-1}$ by using the opposite branch of the square root we would have parametrized the same subset of $\overset{\circ}{S}$ because of the identity $\widetilde{\varphi}_j^{-1}(z) = \varphi_j^{-1}(-z)$. A direct computation shows that $\varphi \circ \varphi_j^{-1}(z) = z^2 + a_j$.

One can give a simple argument to show that $\overset{\circ}{S}$ is connected. Whenever x describes a path joining x_0 to a_j, the map

$$x \longmapsto \left(x, \sqrt{\prod_{i=1}^{2g+1} (x - a_k)} \right)$$

where the root is determined by analytic continuation, describes a path in $\overset{\circ}{S}$ joining $\left(x_0, \sqrt{\prod_{k=1}^{2g+1} (x_0 - a_k)} \right)$ to $(a_j, 0)$. A precise definition of what we mean by the term *analytic continuation* is given in Section 2.9. It corresponds in this case to the simple idea of choosing the branch of the square root along the path in a way that makes the process continuous.

A compact surface S can be obtained out of $\overset{\circ}{S}$, in the same way we constructed the surface $\widehat{\mathbb{C}}$ by adding one abstract point to the complex plane $\mathbb{C}$, see Example 1.5. We also denote this additional point by ∞, and we define a coordinate neighbourhood as follows:

- A parametrization of a neighbourhood of $P_0 = \infty$ is given by

$$\psi^{-1}(z) = \begin{cases} \left(\dfrac{1}{z^2}, \dfrac{1}{z^{2g+1}} \sqrt{\prod_{k=1}^{2g+1} (1 - a_k z^2)} \right) & \text{if } 0 < |z| < \varepsilon \\ \infty \quad \text{if } z = 0 \end{cases}$$

This case is similar to that of φ_j in the sense that simply writing $1/z$ in the first coordinate would not work, and also because the choice of the branch of the square root is irrelevant. This way we have $\varphi \circ \psi^{-1}(z) = 1/z^2$, which is clearly a holomorphic function (its domain of definition does not contain $z = 0$). No computation is needed to check the compatibility of φ_j and ψ, since their domains of definition can be chosen to be disjoint sets.

Finally, we observe that the Riemann surface $S = \overset{\circ}{S} \cup \{\infty\}$ so constructed is compact, since we can decompose S as the union of two compact sets, namely

$$\left\{(x,y) \in \overset{\circ}{S} : \ |x| \leq 1/\varepsilon\right\} \cup \left(\left\{(x,y) \in \overset{\circ}{S} : \ |x| \geq 1/\varepsilon\right\} \cup \{\infty\}\right)$$

The first one is compact because it is a bounded closed subset of $\mathbb{C}^2$, and the second one because it agrees with $\psi^{-1}\left(\overline{\mathbb{D}(0,\sqrt{\varepsilon})}\right)$.

Example 1.8 (Hyperelliptic curves II) The compact surfaces constructed out of the algebraic curves $y^2 = \prod_{k=1}^{2g+2}(x - a_k)$ are also called hyperelliptic curves. The charts are defined as in the previous example, and only the compactification process is slightly different.

We can add a point ∞_1 with a parametrization

$$\psi_1^{-1}(z) = \begin{cases} \left(\dfrac{1}{z}, \dfrac{\sqrt{\prod_{k=1}^{2g+2}(1 - a_k z)}}{z^{g+1}}\right) & \text{if } 0 < |z| < \varepsilon \\ \infty_1 \quad \text{if } z = 0 & \end{cases}$$

The branch of the square root turns out to be relevant now. In fact, if in the expression above the symbol $\sqrt{w}$ denotes a given holomorphic choice of the square root in a neighbourhood of the point $w = 1$, then we can define a second mapping

$$\psi_2^{-1}(z) = \begin{cases} \left(\dfrac{1}{z}, \dfrac{-\sqrt{\prod_{k=1}^{2g+2}(1 - a_k z)}}{z^{g+1}}\right) & \text{if } 0 < |z| < \varepsilon \\ \infty_2 \quad \text{if } z = 0 & \end{cases}$$

to be the parametrization of a second abstract point which we have denoted by ∞_2. If we choose ε small enough in the definition of ψ_1 and ψ_2, both mappings parametrize disjoint sets, and this justifies that $\infty_2 \neq \infty_1$. In this respect the situation here is different from

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the previous example. There, the image of the punctured discs $\{0 < |z| < \varepsilon\}$ by the two possible parametrizations corresponding to the two choices of the square root would be the same open set U_ε, and so any two neighbourhoods $U_{\varepsilon_1} \cup \{\infty_1\}$ of ∞_1 and $U_{\varepsilon_2} \cup \{\infty_2\}$ of ∞_2 would have a non-empty intersection, hence the resulting space would not be Hausdorff, i.e. it would not even be a topological surface.

Remark 1.9 We anticipate that Examples 1.7 and 1.8 produce essentially the same collection of Riemann surfaces (see Example 1.83). These are called *hyperelliptic curves* when $g > 1$ and *elliptic curves* if $g = 1$.

Example 1.10 (Fermat curves) A similar technique produces a compact Riemann surface associated to the curve $x^d + y^d = 1$. Now we start with

$$\mathring{S} = \left\{(x, y) \in \mathbb{C}^2 : x^d + y^d = 1\right\}$$

and denote by ξ_d a chosen primitive root of unity of order d, as for example $\xi_d = e^{2\pi i/d}$. In what follows $\sqrt[d]{w}$ will stand for a determined holomorphic choice of the complex d-th root in a neighbourhood of the non-zero value in question in each case.

The Riemann surface structure in $\mathring{S}$ is given by the following charts:

- If $P_0 = \left(x_0, \xi_d^j \sqrt[d]{1 - x_0^d}\right)$ with $x_0 \neq \xi_d^k$ for $k = 1, \ldots, d$, we take

  $$\varphi^{-1}(z) = \left(z, \xi_d^j \sqrt[d]{1 - z^d}\right)$$

  which is defined in the disc $\{|z - x_0| < \varepsilon\}$.
- If $P_0 = \left(\xi_d^j, 0\right)$, we take

  $$\varphi_j^{-1}(z) = \left(z^d + \xi_d^j, z \sqrt[d]{-\prod \left(z^d + \xi_d^j - \xi_d^k\right)}\right)$$

  where $|z| < \varepsilon$, and the product runs for $k \neq j$.

We can now compactify $\mathring{S}$, as in the previous examples, by adding d *points at infinity*, say $\infty_1, \ldots, \infty_d$:

- For $P_0 = \infty_j$, we take the parametrization

$$\psi_j^{-1}(z) = \begin{cases} \left(\dfrac{1}{z}, \dfrac{\xi_d^j \sqrt[d]{(1-z^d)}}{z}\right), & \text{if } |z| < \varepsilon \\ \infty_j, & \text{if } z = 0 \end{cases}$$

Note again that if $|z| < 1$ then the mappings ψ_j^{-1} have disjoint images, hence the d points we have added to $\overset{\circ}{S}$ are in fact distinct.

Example 1.11 (p-gonal curves) Consider the algebraic curve

$$y^p = (x - a_1)^{m_1} \cdots (x - a_r)^{m_r}$$

where $1 \leq m_i < p$. Now the parametrizations that make this curve into a Riemann surface are

$$\varphi^{-1}(z) = \left(z, \sqrt[p]{\prod_i (z - a_i)^{m_i}}\right)$$

for a neighbourhood of a point (x, y) with $x \neq a_i$ and

$$\varphi_i^{-1}(z) = \left(z^p + a_i, t^{m_i} \sqrt[p]{\prod_{j \neq i} (z^p + a_i - a_j)^{m_j}}\right)$$

for a neighbourhood of $(a_i, 0)$.

Along the lines of the previous examples one can add a point at infinity if $\sum m_i$ is prime to p (or p points otherwise) to obtain a compact Riemann surface.

Throughout the text we will turn our attention several times (see Proposition 1.44 and Section 2.5.2) to a notorious Riemann surface of this type, which is the one associated to the algebraic equation $y^7 = x(x-1)^2$. This is usually known as *Klein's curve of genus three*.

The next examples show how to construct some Riemann surfaces as quotient spaces.

Example 1.12 (The cylinder) $\mathbb{C}/\mathbb{Z}$ usually denotes the quotient set of the complex plane by the following equivalence relation. Two points in $\mathbb{C}$ are in the same class when they differ by an integer number. Thus, a point w is equivalent to all points $w+n$ with $n \in \mathbb{Z}$, i.e. to all its images by the translations $z \mapsto z+n$. The topology to be considered is, of course, the quotient topology.

If $U \subset \mathbb{C}$ is an open set such that no pair of its points belong to the same equivalence class, that is if the canonical projection $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ is injective when restricted to U, then we define a coordinate chart by $\varphi_U := (\pi_{|_U})^{-1} : \pi(U) \to U$.

Suppose that two such coordinate neighbourhoods $\pi(U)$ and $\pi(V)$ have non-empty intersection. Let $P \in \pi(U) \cap \pi(V)$, that is $P = \pi(z_1) = \pi(z_2)$ with $z_1 \in U$ and $z_2 \in V$. Then we have $z_2 = z_1 + m$ for some $m \in \mathbb{Z}$. Therefore, in $\varphi_U(\pi(U) \cap \pi(V)) \subset U$, the transition function takes the form $\varphi_V \circ \varphi_U^{-1}(z) = z + m(z)$, with $m(z) \in \mathbb{Z}$. But $\varphi_V \circ \varphi_U^{-1}$ is a continuous function, hence $m(z)$ is locally constant. We thus see that the restriction of the transition functions to each connected component of their domain of definition is a translation by an integer m.

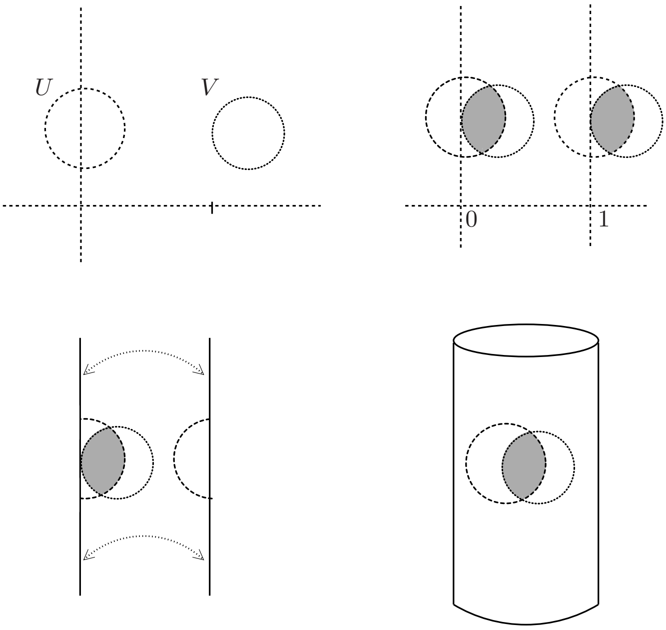

Fig. 1.2. $\mathbb{C}/\mathbb{Z}$ is topologically a cylinder.

The vertical strip $F = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$ is a *fundamental domain*, i.e. a subset of $\mathbb{C}$ containing at least one representative of every equivalence class and exactly one except in the boundary (cf. Definition 2.5). The surface $\mathbb{C}/\mathbb{Z}$ is obtained after glueing the two straight lines at the boundary of F. It is therefore a cylinder (see Figure 1.2).

Example 1.13 (A complex torus) As in the previous example, let us identify every $w \in \mathbb{C}$ with its images under all translations by Gaussian integers, that is complex numbers whose real and imaginary parts are both integer numbers. The classes for this equivalence relation are $[w] = \{w + n + mi,\ n, m \in \mathbb{Z}\}$. The corresponding quotient set

$$\mathbb{C}/\Lambda, \quad \text{where} \quad \Lambda = \mathbb{Z} \oplus \mathbb{Z}i$$

can be described as in the previous example. Arguing as above we see that the transition functions take in each connected component the form

$$\varphi_V \circ \varphi_U^{-1}(z) = z + \lambda,\ \lambda \in \Lambda$$

and the parallelogram in Figure 1.3 can be chosen as fundamental domain. Opposite sides are identified to form the quotient space, thus $\mathbb{C}/\Lambda$ is topologically a torus.

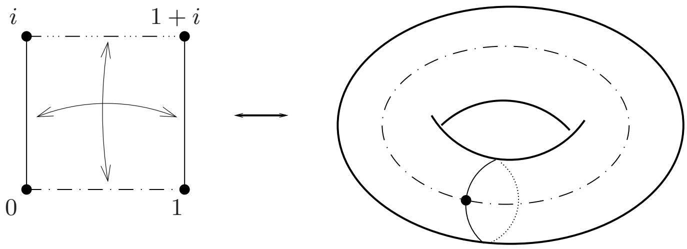

Fig. 1.3. $\mathbb{C}/\Lambda$ is topologically a torus.

Example 1.14 A Riemannian structure (i.e. a metric) on an orientable surface induces a Riemann surface structure whose charts (U_i, φ_i) are conformal bijections preserving orientation ([dC76]).

As the transition functions $\varphi_j \circ \varphi_i^{-1}$ must also be (orientation-preserving) conformal bijections between open subsets of $\mathbb{C}$, they are holomorphic ([Ahl78]).

1.1.2 Morphisms of Riemann surfaces

Complex analysis can be defined on a Riemann surface, the concept of holomorphy being the obvious one.

Definition 1.15 Let S be a Riemann surface and $f : S \longrightarrow \mathbb{C}$ a function. We say that f is *holomorphic* (resp. *meromorphic*) if, for any coordinate function φ, the function $f \circ \varphi^{-1}$ is holomorphic (resp. meromorphic) in the usual sense of complex analysis. The set of meromorphic functions on S is a field, which shall be denoted by $\mathcal{M}(S)$.

The same idea behind the previous definition, i.e. using local coordinates, can be applied to consider functions on a Riemann surface which take more general values than complex numbers. In fact, one can replace the target space $\mathbb{C}$ by any other Riemann surface.

Definition 1.16 A *morphism* between two Riemann surfaces S and S' is a continuous mapping $f : S \longrightarrow S'$ such that $\varphi' \circ f \circ \varphi^{-1}$ is a holomorphic function for every choice of coordinates φ in S and φ' in S' for which the composition makes sense. We will denote by $\mathrm{Mor}(S, S')$ the set of morphisms from S to S'.

Bijective morphisms are called *isomorphisms* and isomorphisms from a surface to itself are called *automorphisms*. The set of automorphisms of a given Riemann surface S forms a group. We shall denote it by $\mathrm{Aut}(S)$.

Remark 1.17 Let $f : S \longrightarrow S'$ be a non-constant morphism between connected compact Riemann surfaces. Since non-constant holomorphic maps are open maps, the image $f(S)$ must be simultaneously open and closed in S', and therefore equal to the whole S'. Thus, f is surjective.

Example 1.18 $\mathbb{H}$ and $\mathbb{D}$ are isomorphic Riemann surfaces via the