Aggregation Functions

Aggregation is the process of combining several numerical values into a single representative value, and an aggregation function performs this operation. These functions arise wherever aggregating information is important: applied and pure mathematics (probability, statistics, decision theory, functional equations), operations research, computer science, and many applied fields (economics and finance, pattern recognition and image processing, data fusion, etc.).

This readable book provides a comprehensive, rigorous and self-contained exposition of aggregation functions. Classes of aggregation functions covered include triangular norms and conorms, copulas, means and averages, and those based on nonadditive integrals. The properties of each method, as well as their interpretation and analysis, are studied in depth, together with construction methods and practical identification methods. Special attention is given to the nature of scales on which values to be aggregated are defined (ordinal, interval, ratio, bipolar). It is an ideal introduction for graduate students and a unique resource for researchers.

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M.G.

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J.-L.M.

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R.M.

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Contents

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Frontmatter
More information

Figures

Tables

Preface

The process of combining several numerical values into a single representative one is called *aggregation*, and the numerical function performing this process is called an *aggregation function*. This simple definition demonstrates the size of the field of application of aggregation: applied mathematics (e.g., probability, statistics, decision theory), computer sciences (e.g., artificial intelligence, operations research), as well as many applied fields (economics and finance, pattern recognition and image processing, data fusion, multicriteria decision aid, automated reasoning, etc.).

Although the history of aggregation is probably as old as mathematics (think of the arithmetic mean), its existence has remained underground till only recently, and its utilization rather intuitive and hardly formalized. The rapid growth of the above-mentioned application fields, largely due to the arrival of computers, has made necessary the establishment of a sound theoretical basis for aggregation functions. Hence, since the 1980s, aggregation functions have become a genuine research field, rapidly developing, but in a rather scattered way since aggregation functions are rooted in many different fields. Indeed, most of the results were disseminated in various journals or specialized books, where usually only one specific class of aggregation functions devoted to one specific domain is discussed.

Actually, in these early years of the twenty-first century, a substantial amount of literature is already available, many significant results have been found (such as characterizations of various families of aggregation functions), and many connections have been made with either related fields or former work (such as triangular norms in probabilistic metric spaces, theory of means and averages, etc.). Yet for the researcher as well as for the practitioner, this abundance of literature, because it is scattered in many domains, is more a handicap than an advantage, and there is a real lack of a unified and complete view of aggregation functions, where one could find the most important concepts and results presented in a clear and rigorous way.

This book has been written with the intention of filling this gap: it offers a full, comprehensive, rigorous, and unified treatment of aggregation functions. Our main motivation has been to bring a unified viewpoint of the aggregation problem, and to provide an abstract mathematical presentation and analysis of aggregation functions used in various disciplines, without referring explicitly to a given domain. The book also provides a unified terminology and notation.

To reach this aim, we have tried to follow as closely as possible the following guidelines. First, by contrast to the style of many handbooks, the chapters are not a collection of definitions, facts and assertions without proof, but we have maintained a straight, visible and logical line in our discourse, avoiding anecdotal details. Second, our aim was not to be exhaustive, citing every latest advance in the field, but to be selective, and put material into a historical perspective. As far as possible, we have tried to provide the original references. Third, the presentation is mathematical and rigorous, avoiding jargon and inherent imprecision from the various applied domains where aggregation functions are used (often under different names such as aggregation operators, merging functions, connectives, etc.), but keeping as far as possible the standard terminology of mathematics. This is the only way to make the book usable by every researcher or practitioner in every field. As far as possible, every result is given with its proof, unless the proof is long and requires extra material. In this case, a reference to the proof is always given.

The book is intended primarily for researchers and graduate students in applied mathematics and computer sciences, secondarily for practitioners in, for example, decision making, optimization, economics and finance, artificial intelligence, data fusion, computer vision, etc. It could also be used as a textbook for graduate students in applied mathematics and computer sciences. The reader of the book is assumed to have the basic knowledge of a graduate student in algebra and analysis.

The table of contents has been detailed. The main theoretical corpus is given in Chapters 2 to 5. Additional theoretical material is given in Chapters 6 to 9, while Chapters 10 and 11 are more practically oriented. In Chapters 2 to 5, as far as possible, most of the results are given with proof. Due to space limitations and forest saving, it has not been possible to maintain this philosophy in the second part of the book, which has too broad a scope.

– *Chapter 1: Introduction*
  The general idea of an aggregation function is presented, and the scope of the book is defined. After giving some basic examples and definitions, the conventional notation for the whole book is presented.

– *Chapter 2: Properties for aggregation*
  This important chapter defines the basic possible properties for aggregation functions. They are divided into elementary mathematical properties (monotonicity, continuity, symmetry, etc.), grouping properties (associativity, decomposability, etc.), scale invariance (ratio, difference, interval, ordinal scales), and various other properties (neutral and annihilator elements, additivity, etc.).

– *Chapter 3: Conjunctive and disjunctive aggregation functions*
  Conjunctive (respectively, disjunctive) aggregation functions are those functions

acting like a logical "and" (respectively, a logical "or"). In this chapter, full development is given for conjunctive aggregation functions. Disjunctive aggregation functions are merely obtained by duality. A large section is devoted to triangular norms (t-norms for short): different families, continuous Archimedean t-norms, additively generated t-norms, ordinal sums, etc. Another important section is devoted to copulas, well known in probability theory. Two other sections present uninorms and nullnorms (combinations of t-norms and t-conorms).

– *Chapter 4: Means and averages*
  This chapter develops perhaps the best known family of aggregation functions, with a long history. The concepts of means and average functions, as well as their relationships, are first presented in full generality. Then main subclasses of means, such as quasi-arithmetic ones, some of their special cases, and some of their generalizations are presented. A section is then devoted to means constructed from the associativity property and another one to those means constructed from a mean value property, such as Cauchy means. A section also concerns some construction methods. Finally, the last section deals with extended means constructed from weight triangles.
– *Chapter 5: Aggregation functions based on nonadditive integrals*
  Considering nonadditive integrals (e.g., the Choquet integral) in the discrete finite case defines a new class of aggregation functions, in which interest developed in the 1980s. Nonadditive integrals are defined with respect to capacities (nonadditive monotone measures), and in particular generalize the notion of expected value. A first section defines capacities, their properties and related notions. An important section is devoted to the Choquet integral, since this is the most representative of nonadditive integrals, possessing many appealing properties. Then the case of the Sugeno integral is presented, and finally other families of nonadditive integrals.
– *Chapter 6: Construction methods*
  This chapter gives some means to create new aggregation functions from existing ones. The main operations to do this are transformation, composition, introduction of weights on variables, ordinal sums, and various other means (idempotization, etc.). Also optimization tasks yielding aggregation functions are discussed.
– *Chapter 7: Aggregation on specific scale types*
  This chapter addresses the important concern of choosing appropriate aggregation functions by taking into account the scale types of the input and output variables. The scale type of a variable is defined by a class of admissible transformations, such as that from grams to pounds or degrees Fahrenheit to degrees centigrade, that change the scale into another acceptable scale. We describe the aggregation functions that are meaningful when considering ratio, difference, interval, and log-ratio scales.
– *Chapter 8: Aggregation on ordinal scales*
  On ordinal scales, all usual arithmetic operations become meaningless in the

measurement theoretical point of view, and allowed operations are more or less limited to comparisons and projections. We investigate which aggregation functions are meaningful on ordinal scales.

- *Chapter 9: Aggregation on bipolar scales*
  Most aggregation functions are defined on the $[0, 1]$ interval (unipolar scale). This chapter analyzes how to extend them to the interval $[-1, 1]$ (bipolar scale), that is, to perform a kind of symmetrization with respect to 0 while keeping properties of the aggregation function. This nontrivial problem is motivated essentially by decision making, where most often bipolar scales are more suitable than unipolar ones.
- *Chapter 10: Behavioral analysis of aggregation functions*
  This chapter gives various ways to understand, analyze and quantify the "behavior" of an aggregation function, that is, how the output of the function behaves with respect to its variables. This is done through various indices and values (like the expected value), which in some sense constitutes the identity card of the aggregation function.
- *Chapter 11: Identification of aggregation functions*
  An important topic in practice is how to choose a suitable aggregation function. Chapters 2 to 5 and Chapter 10 provide the keys to selecting the suitable family of aggregation functions and to understanding its behavior, but a precise identification (i.e., what is the value of the parameter(s)?) is not always possible. This chapter gives various ways to identify aggregation functions from data. This often reduces to solving an optimization problem, most of the time a least squares regression problem under constraints.
- *Appendix A: Aggregation of infinitely many arguments*
  This appendix explores the rather unexpected consequences of defining an aggregation function with an infinite (either countable or uncountable) number of arguments.
- *Appendix B: Examples and applications*
  A short description is given with references to the main fields of application of aggregation functions, namely in decision making, data fusion, and artificial intelligence. A last section details an application to the mixture of uncertainty measures.

The genesis of the book goes back to the summer of 2002, on the shores of Lake Annecy, a charming place in the Alps in the South of France. We were there together for the IPMU Congress, and inspired by the beauty of the landscape, we decided to start the great adventure of writing a book on aggregation functions. During the next six years, we exchanged hundreds of emails, and took advantage as far as possible of many congresses, workshops, and projects to meet, visit each other, discuss the book, and incidentally see many other nice landscapes. We started as colleagues in mathematics and computer science, and finished as close friends, having experienced and learnt a lot, apart from mathematics, about ourselves and each other.

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The whole manuscript was typeset in LaTeX 2_ε; most of the figures were drawn using Mathematica 5.2 and `pstricks`.