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Random Walk: A Modern Introduction

Random walks are stochastic processes formed by successive summation of independent, identically distributed random variables and are one of the most studied topics in probability theory. This contemporary introduction evolved from courses taught at Cornell University and the University of Chicago by the first author, who is one of the most highly regarded researchers in the field of stochastic processes. This text meets the need for a modern reference to the detailed properties of an important class of random walks on the integer lattice.

It is suitable for probabilists, mathematicians working in related fields, and for researchers in other disciplines who use random walks in modeling.

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Preface

Random walk – the stochastic process formed by successive summation of independent, identically distributed random variables – is one of the most basic and well-studied topics in probability theory. For random walks on the integer lattice $\mathbb{Z}^d$, the main reference is the classic book by Spitzer (1976). This text considers only a subset of such walks, namely those corresponding to increment distributions with zero mean and finite variance. In this case, one can summarize the main result very quickly: the central limit theorem implies that under appropriate rescaling the limiting distribution is normal, and the functional central limit theorem implies that the distribution of the corresponding path-valued process (after standard rescaling of time and space) approaches that of Brownian motion.

Researchers who work with perturbations of random walks, or with particle systems and other models that use random walks as a basic ingredient, often need more precise information on random walk behavior than that provided by the central limit theorems. In particular, it is important to understand the size of the error resulting from the approximation of random walk by Brownian motion. For this reason, there is need for more detailed analysis. This book is an introduction to the random walk theory with an emphasis on the error estimates. Although "mean zero, finite variance" assumption is both necessary and sufficient for normal convergence, one typically needs to make stronger assumptions on the increments of the walk in order to obtain good bounds on the error terms.

This project was embarked upon with an idea of writing a book on the simple, nearest neighbor random walk. Symmetric, finite range random walks gradually became the central model of the text. This class of walks, while being rich enough to require analysis by general techniques, can be studied without much additional difficulty. In addition, for some of the results, in particular, the local central limit theorem and the Green's function estimates, we have extended the discussion to include other mean zero, finite variance walks, while indicating the way in which moment conditions influence the form of the error.

The first chapter is introductory and sets up the notation. In particular, there are three main classes of irreducible walk in the integer lattice $\mathbb{Z}^d$ — $\mathcal{P}_d$ (symmetric, finite range), $\mathcal{P}'_d$ (aperiodic, mean zero, finite second moment), and $\mathcal{P}^*_d$ (aperiodic with no other assumptions). Symmetric random walks on other integer lattices such as the triangular lattice can also be considered by taking a linear transformation of the lattice onto $\mathbb{Z}^d$.

The local central limit theorem (LCLT) is the topic for Chapter 2. Its proof, like the proof of the usual central limit theorem, is done by using Fourier analysis to express the probability of interest in terms of an integral, and then estimating the integral. The error estimates depend strongly on the number of finite moments of the corresponding increment distribution. Some important corollaries are proved in Section 2.4; in particular, the fact that aperiodic random walks starting at different points can be coupled so that with probability $1 - O(n^{-1/2})$ they agree for all times greater than n is true for any aperiodic walk, without any finite moment assumptions. The chapter ends by a more classical, combinatorial derivation of LCLT for simple random walk using Stirling's formula, while again keeping track of error terms.

Brownian motion is introduced in Chapter 3. Although we would expect a typical reader to be familiar already with Brownian motion, we give the construction via the dyadic splitting method. The estimates for the modulus of continuity are also given. We then describe the Skorokhod method of coupling a random walk and a Brownian motion on the same probability space, and give error estimates. The dyadic construction of Brownian motion is also important for the dyadic coupling algorithm of Chapter 7.

Green's function and its analog in the recurrent setting, the potential kernel, are studied in Chapter 4. One of the main tools in the potential theory of random walk is the analysis of martingales derived from these functions. Sharp asymptotics at infinity for Green's function are needed to take full advantage of the martingale technique. We use the sharp LCLT estimates of Chapter 2 to obtain the Green's function estimates. We also discuss the number of finite moments needed for various error asymptotics.

Chapter 5 may seem somewhat out of place. It concerns a well-known estimate for one-dimensional walks called the gambler's ruin estimate. Our motivation for providing a complete self-contained argument is twofold. First, in order to apply this result to all one-dimensional projections of a higher dimensional walk simultaneously, it is important to show that this estimate holds for non-lattice walks uniformly in few parameters of the distribution (variance, probability of making an order 1 positive step). In addition, the argument introduces the reader to a fairly general technique for obtaining the overshoot estimates. The final two sections of this chapter concern variations of one-dimensional walk that arise naturally in the arguments for estimating probabilities of hitting (or avoiding) some special sets, for example, the half-line.

In Chapter 6, the classical potential theory of the random walk is covered in the spirit of Spitzer (1976) and Lawler (1996) (and a number of other sources). The difference equations of our discrete space setting (that in turn become matrix equations on finite sets) are analogous to the standard linear partial differential equations of (continuous) potential theory. The closed form of the solutions is important, but we emphasize here the estimates on hitting probabilities that one can obtain using them. The martingales derived from Green's function are very important in this analysis, and again special care is given to error terms. For notational ease, the discussion is restricted here to symmetric walks. In fact, most of the results of this chapter hold for nonsymmetric walks, but in this case one must distinguish between the "original" walk and the "reversed" walk, i.e. between an operator and its adjoint. An implicit exercise for a dedicated student would be to redo this entire chapter for nonsymmetric walks, changing the statements of the propositions as necessary. It would be more work to relax the finite range assumption, and the moment conditions would become a crucial component of the analysis in this general setting. Perhaps this will be a topic of some future book.

Chapter 7 discusses a tight coupling of a random walk (that has a finite exponential moment) and a Brownian motion, called the dyadic coupling or KMT or Hungarian coupling, originated in Kómlos *et al*. (1975a, b). The idea of the coupling is very natural (once explained), but hard work is needed to prove the strong error estimate. The sharp LCLT estimates from Chapter 2 are one of the key points for this analysis.

In bounded rectangles with sides parallel to the coordinate directions, the rate of convergence of simple random walk to Brownian motion is very fast. Moreover, in this case, exact expressions are available in terms of finite Fourier sums. Several of these calculations are done in Chapter 8.

Chapter 9 is different from the rest of this book. It covers an area that includes both classical combinatorial ideas and topics of current research. As has been gradually discovered by a number of researchers in various disciplines (combinatorics, probability, statistical physics) several objects inherent to a graph or network are closely related: the number of spanning trees, the determinant of the Laplacian, various measures on loops on the trees, Gaussian free field, and loop-erased walks. We give an introduction to this theory, using an approach that is focused on the (unrooted) random walk loop measure, and that uses Wilson's algorithm (1996) for generating spanning trees.

The original outline of this book put much more emphasis on the path-intersection probabilities and the loop-erased walks. The final version offers only a general introduction to some of the main ideas, in the last two chapters. On the one hand, these topics were already discussed in more detail in Lawler (1996), and on the other, discussing the more recent developments in the area would require familiarity with Schramm–Loewner evolution, and explaining this would take us too far from the main topic.

Most of the content of this text (the first eight chapters in particular) are well-known classical results. It would be very difficult, if not impossible, to give a detailed and complete list of references. In many cases, the results were obtained in several places at different occasions, as auxiliary (technical) lemmas needed for understanding some other model of interest, and were therefore not particularly noticed by the community. Attempting to give even a reasonably fair account of the development of this subject would have inhibited the conclusion of this project. The bibliography is therefore restricted to a few references that were used in the writing of this book. We refer the reader to Spitzer (1976) for an extensive bibliography on random walk, and to Lawler (1996) for some additional references.

This book is intended for researchers and graduate students alike, and a considerable number of exercises is included for their benefit. The appendix consists of various results from probability theory that are used in the first eleven chapters but are, however, not really linked to random walk behavior. It is assumed that the reader is familiar with the basics of measure-theoretic probability theory.

♣ The book contains quite a few remarks that are separated from the rest of the text by this typeface. They are intended to be helpful heuristics for the reader, but are not used in the actual arguments.

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