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1
Introduction

1.1 Basic definitions

We will define the random walks that we consider in this book. We focus our attention on random walks in $\mathbb{Z}^d$ that have bounded symmetric increment distributions, although we occasionally discuss results for wider classes of walk. We also impose an irreducibility criterion to guarantee that all points in the lattice $\mathbb{Z}^d$ can be reached.

We start by setting some basic notation. We use x, y, z to denote points in the integer lattice $\mathbb{Z}^d = \{(x^1, \ldots, x^d) : x^j \in \mathbb{Z}\}$. We use superscripts to denote components and we use subscripts to enumerate elements. For example, $x_1, x_2, \ldots$ represents a sequence of points in $\mathbb{Z}^d$, and the point x_j can be written in component form $x_j = (x_j^1, \ldots, x_j^d)$. We write $\mathbf{e}_1 = (1, 0, \ldots, 0), \ldots, \mathbf{e}_d = (0, \ldots, 0, 1)$ for the standard basis of unit vectors in $\mathbb{Z}^d$. The prototypical example is (discrete time) simple random walk starting at $x \in \mathbb{Z}^d$. This process can be considered either as a sum of a sequence of independent, identically distributed random variables

$$S_n = x + X_1 + \cdots + X_n$$

where $\mathbb{P}\{X_j = \mathbf{e}_k\} = \mathbb{P}\{X_j = -\mathbf{e}_k\} = 1/(2d), k = 1, \ldots, d$, or it can be considered as a Markov chain with state space $\mathbb{Z}^d$ and transition probabilities

$$\mathbb{P}\{S_{n+1} = z \mid S_n = y\} = \frac{1}{2d}, \quad z - y \in \{\pm \mathbf{e}_1, \ldots \pm \mathbf{e}_d\}.$$

We call $V = \{x_1, \ldots, x_l\} \subset \mathbb{Z}^d \setminus \{0\}$ a *(finite) generating set* if each $y \in \mathbb{Z}^d$ can be written as $k_1 x_1 + \cdots + k_l x_l$ for some $k_1, \ldots, k_l \in \mathbb{Z}$. We let $\mathcal{G}$ denote the collection of generating sets V with the property that if $x = (x^1, \ldots, x^d) \in V$, then the first nonzero component of x is positive. An example of such a set is

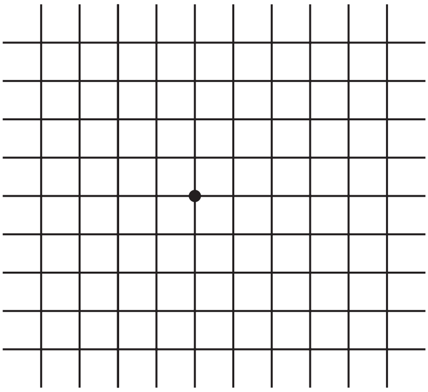

Figure 1.1 The square lattice $\mathbb{Z}^2$

$\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$. A *(finite range, symmetric, irreducible) random walk* is given by specifying a $V = \{x_1, \ldots, x_l\} \in \mathcal{G}$ and a function $\kappa : V \to (0, 1]$ with $\kappa(x_1) + \cdots + \kappa(x_l) \leq 1$. Associated to this is the symmetric probability distribution on $\mathbb{Z}^d$

$$p(x_k) = p(-x_k) = \frac{1}{2}\kappa(x_k), \qquad p(0) = 1 - \sum_{x \in V} \kappa(x).$$

We let $\mathcal{P}_d$ denote the set of such distributions p on $\mathbb{Z}^d$ and $\mathcal{P} = \cup_{d \geq 1} \mathcal{P}_d$. Given p, the corresponding random walk S_n can be considered as the time-homogeneous Markov chain with state space $\mathbb{Z}^d$ and transition probabilities

$$p(y, z) := \mathbb{P}\{S_{n+1} = z \mid S_n = y\} = p(z - y).$$

We can also write

$$S_n = S_0 + X_1 + \cdots + X_n$$

where $X_1, X_2, \ldots$ are independent random variables, independent of S_0, with distribution p. (Most of the time, we will choose S_0 to have a trivial distribution.) We will use the phrase $\mathcal{P}$-walk or $\mathcal{P}_d$-walk for such a random walk. We will use the term *simple random walk* for the particular p with

$$p(\mathbf{e}_j) = p(-\mathbf{e}_j) = \frac{1}{2d}, \quad j = 1, \ldots, d.$$

We call p the *increment distribution* for the walk. Given that $p \in \mathcal{P}$, we write p_n for the n-step distribution

$$p_n(x, y) = \mathbb{P}\{S_n = y \mid S_0 = x\}$$

and $p_n(x) = p_n(0, x)$. Note that $p_n(\cdot)$ is the distribution of $X_1 + \cdots + X_n$ where $X_1, \ldots, X_n$ are independent with increment distribution p.

♣ In many ways the main focus of this book is simple random walk, and a first-time reader might find it useful to consider this example throughout. We have chosen to generalize this slightly, because it does not complicate the arguments much and allows the results to be extended to other examples. One particular example is simple random walk on other regular lattices such as the planar triangular lattice. In Section 1.3, we show that walks on other d-dimensional lattices are isomorphic to p-walks on $\mathbb{Z}^d$.

If $S_n = (S_n^1, \ldots, S_n^d)$ is a $\mathcal{P}$-walk with $S_0 = 0$, then $\mathbb{P}\{S_{2n} = 0\} > 0$ for every even integer n; this follows from the easy estimate $\mathbb{P}\{S_{2n} = 0\} \geq [\mathbb{P}\{S_2 = 0\}]^n \geq p(x)^{2n}$ for every $x \in \mathbb{Z}^d$. We will call the walk *bipartite* if $p_n(0, 0) = 0$ for every odd n, and we will call it *aperiodic* otherwise. In the latter case, $p_n(0, 0) > 0$ for all n sufficiently large (in fact, for all $n \geq k$ where k is the first odd integer with $p_k(0, 0) > 0$). Simple random walk is an example of a bipartite walk since $S_n^1 + \cdots + S_n^d$ is odd for odd n and even for even n. If p is bipartite, then we can partition $\mathbb{Z}^d = (\mathbb{Z}^d)_e \cup (\mathbb{Z}^d)_o$ where $(\mathbb{Z}^d)_e$ denotes the points that can be reached from the origin in an even number of steps and $(\mathbb{Z}^d)_o$ denotes the set of points that can be reached in an odd number of steps. In algebraic language, $(\mathbb{Z}^d)_e$ is an additive subgroup of $\mathbb{Z}^d$ of index 2 and $(\mathbb{Z}^d)_o$ is the nontrivial coset. Note that if $x \in (\mathbb{Z}^d)_o$, then $(\mathbb{Z}^d)_o = x + (\mathbb{Z}^d)_e$.

♣ It would suffice and would perhaps be more convenient to restrict our attention to aperiodic walks. Results about bipartite walks can easily be deduced from them. However, since our main example, simple random walk, is bipartite, we have chosen to allow such p.

If $p \in \mathcal{P}_d$ and $j_1, \ldots, j_d$ are nonnegative integers, the $(j_1, \ldots, j_d)$ moment is given by

$$\mathbb{E}[(X_1^1)^{j_1} \cdots (X_1^d)^{j_d}] = \sum_{x \in \mathbb{Z}^d} (x^1)^{j_1} \cdots (x^d)^{j_d}\, p(x).$$

We let Γ denote the *covariance matrix*

$$\Gamma = \left[\mathbb{E}[X_1^j X_1^k] \right]_{1 \le j,k \le d}.$$

The covariance matrix is symmetric and positive definite. Since the random walk is truly d-dimensional, it is easy to verify (see Proposition 1.1.1 (a)) that the matrix Γ is invertible. There exists a symmetric positive definite matrix Λ such that $\Gamma = \Lambda\,\Lambda^T$ (see Section A.3). There is a (not unique) orthonormal basis $u_1, \ldots, u_d$ of $\mathbb{R}^d$ such that we can write

$$\Gamma x = \sum_{j=1}^{d} \sigma_j^2\,(x \cdot u_j)\,u_j, \qquad \Lambda x = \sum_{j=1}^{d} \sigma_j\,(x \cdot u_j)\,u_j.$$

If X_1 has covariance matrix $\Gamma = \Lambda\,\Lambda^T$, then the random vector $\Lambda^{-1} X_1$ has covariance matrix I.

For future use, we define norms $\mathcal{J}^*, \mathcal{J}$ by

$$\mathcal{J}^*(x)^2 = |x \cdot \Gamma^{-1} x| = |\Lambda^{-1} x|^2 = \sum_{j=1}^{d} \sigma_j^{-2}\,(x \cdot u_j)^2, \qquad \mathcal{J}(x) = d^{-1/2}\,\mathcal{J}^*(x). \tag{1.1}$$

If $p \in \mathcal{P}_d$,

$$\mathbb{E}[\mathcal{J}(X_1)^2] = \frac{1}{d}\,\mathbb{E}[\mathcal{J}^*(X_1)^2] = \frac{1}{d}\,\mathbb{E}\left[|\Lambda^{-1} X_1|^2\right] = 1.$$

For simple random walk in $\mathbb{Z}^d$,

$$\Gamma = d^{-1}\,I, \qquad \mathcal{J}^*(x) = d^{1/2}\,|x|, \qquad \mathcal{J}(x) = |x|.$$

We will use $\mathcal{B}_n$ to denote the discrete ball of radius n,

$$\mathcal{B}_n = \{x \in \mathbb{Z}^d : |x| < n\},$$

and $\mathcal{C}_n$ to denote the discrete ball under the norm $\mathcal{J}$,

$$\mathcal{C}_n = \{x \in \mathbb{Z}^d : \mathcal{J}(x) < n\} = \{x \in \mathbb{Z}^d : \mathcal{J}^*(x) < d^{1/2}\,n\}.$$

We choose to use $\mathcal{J}$ in the definition of $\mathcal{C}_n$ so that for simple random walk, $\mathcal{C}_n = \mathcal{B}_n$. We will write $R = R_p = \max\{|x| : p(x) > 0\}$ and we will call R the

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range of p. The following is very easy, but it is important enough to state as a proposition.

Proposition 1.1.1 *Suppose that $p \in \mathcal{P}_d$.*

(a) There exists an $\epsilon > 0$ such that for every unit vector $u \in \mathbb{R}^d$,

$$\mathbb{E}[(X_1 \cdot u)^2] \geq \epsilon.$$

(b) If $j_1, \ldots, j_d$ are nonnegative integers with $j_1 + \cdots + j_d$ odd, then

$$\mathbb{E}[(X_1^1)^{j_1} \cdots (X_1^d)^{j_d}] = 0.$$

(c) There exists a $\delta > 0$ such that for all x,

$$\delta\, \mathcal{J}(x) \leq |x| \leq \delta^{-1}\, \mathcal{J}(x).$$

In particular,

$$\mathcal{C}_{\delta n} \subset \mathcal{B}_n \subset \mathcal{C}_{n/\delta}.$$

We note for later use that we can construct a random walk with increment distribution $p \in \mathcal{P}$ from a collection of independent one-dimensional simple random walks and an independent multinomial process. To be more precise, let $V = \{x_1, \ldots, x_l\} \in \mathcal{G}$ and let $\kappa : V \to (0,1]^l$ be as in the definition of $\mathcal{P}$. Suppose that on the same probability space we have defined l independent one-dimensional simple random walks $S_{n,1}, S_{n,2}, \ldots, S_{n,l}$ and an independent multinomial process $L_n = (L_n^1, \ldots, L_n^l)$ with probabilities $\kappa(x_1), \ldots, \kappa(x_l)$. In other words,

$$L_n = \sum_{j=1}^{n} Y_j,$$

where $Y_1, Y_2, \ldots$ are independent $\mathbb{Z}^l$-valued random variables with

$$\mathbb{P}\{Y_k = (1, 0, \ldots, 0)\} = \kappa(x_1), \ldots, \mathbb{P}\{Y_k = (0, 0, \ldots, 1)\} = \kappa(x_l),$$

and $\mathbb{P}\{Y_k = (0, 0, \ldots, 0)\} = 1 - [\kappa(x_1) + \cdots + \kappa(x_l)]$. It is easy to verify that the process

$$S_n := x_1\, S_{L_n^1,1} + x_2\, S_{L_n^2,2} + \cdots + x_l\, S_{L_n^l,l} \tag{1.2}$$

has the distribution of the random walk with increment distribution p. Essentially, what we have done is to split the decision as to how to jump at time n into two decisions: first, to choose an element $x_j \in \{x_1, \ldots, x_l\}$ and then to decide whether to move by $+x_j$ or $-x_j$.

1.2 Continuous-time random walk

It is often more convenient to consider random walks in $\mathbb{Z}^d$ indexed by positive real times. Given that V, κ, p as in the previous section, the *continuous-time random walk with increment distribution p* is the continuous-time Markov chain $\tilde{S}_t$ with rates p. In other words, for each $x, y \in \mathbb{Z}^d$,

$$\mathbb{P}\{\tilde{S}_{t+\Delta t} = y \mid \tilde{S}_t = x\} = p(y-x)\,\Delta t + o(\Delta t), \quad y \neq x,$$

$$\mathbb{P}\{\tilde{S}_{t+\Delta t} = x \mid \tilde{S}_t = x\} = 1 - \left[\sum_{y \neq x} p(y-x)\right] \Delta t + o(\Delta t).$$

Let $\tilde{p}_t(x,y) = \mathbb{P}\{\tilde{S}_t = y \mid \tilde{S}_0 = x\}$, and $\tilde{p}_t(y) = \tilde{p}_t(0,y) = \tilde{p}_t(x, x+y)$. Then the expressions above imply that

$$\frac{d}{dt}\tilde{p}_t(x) = \sum_{y \in \mathbb{Z}^d} p(y)\,[\tilde{p}_t(x-y) - \tilde{p}_t(x)].$$

There is a very close relationship between the discrete time and continuous time random walks with the same increment distribution. We state this as a proposition which we leave to the reader to verify.

Proposition 1.2.1 *Suppose that S_n is a (discrete-time) random walk with increment distribution p and N_t is an independent Poisson process with parameter* 1. *Then $\tilde{S}_t := S_{N_t}$ has the distribution of a continuous-time random walk with increment distribution p.*

There are various technical reasons why continuous-time random walks are sometimes easier to handle than discrete-time walks. One reason is that in the continuous setting there is no periodicity. If $p \in \mathcal{P}_d$, then $\tilde{p}_t(x) > 0$ for every $t > 0$ and $x \in \mathbb{Z}^d$. Another advantage can be found in the following proposition which gives an analogous, but nicer, version of (1.2). We leave the proof to the reader.

Proposition 1.2.2 *Suppose that $p \in \mathcal{P}_d$ with generating set $V = \{x_1, \ldots, x_l\}$ and suppose that $\tilde{S}_{t,1}, \ldots, \tilde{S}_{t,l}$ are independent one-dimensional* *continuous-time random walks with increment distribution* $q_1, \ldots, q_l$ *where* $q_j(\pm 1) = p(x_j)$. *Then*

$$\tilde{S}_t := x_1\, \tilde{S}_{t,1} + x_2\, \tilde{S}_{t,2} + \cdots + x_l\, \tilde{S}_{t,l} \tag{1.3}$$

has the distribution of a continuous-time random walk with increment distribution p.

If p is the increment distribution for simple random walk, we call the corresponding walk $\tilde{S}_t$ the *continuous-time simple random walk in* $\mathbb{Z}^d$. From the previous proposition, we see that the coordinates of the continuous-time simple random walk are independent — this is clearly not true for the discrete-time simple random walk. In fact, we get the following. Suppose that $\tilde{S}_{t,1}, \ldots, \tilde{S}_{t,d}$ are independent one-dimensional continuous-time simple random walks. Then,

$$\tilde{S}_t := (\tilde{S}_{t/d,1}, \ldots, \tilde{S}_{t/d,d})$$

is a continuous time simple random walk in $\mathbb{Z}^d$. In particular, if $\tilde{S}_0 = 0$, then

$$\mathbb{P}\{\tilde{S}_t = (y^1, \ldots, y^d)\} = \mathbb{P}\{\tilde{S}_{t/d,1} = y^1\} \cdots \mathbb{P}\{\tilde{S}_{t/d,l} = y^l\}.$$

Remark To verify that a discrete-time process S_n is a random walk with distribution $p \in \mathcal{P}_d$ starting at the origin, it suffices to show for all positive integers $j_1 < j_2 < \cdots < j_k$ and $x_1, \ldots, x_k \in \mathbb{Z}^d$,

$$\mathbb{P}\{S_{j_1} = x_1, \ldots, S_{j_k} = x_k\} = p_{j_1}(x_1)\, p_{j_2 - j_1}(x_2 - x_1) \cdots p_{j_k - j_{k-1}}(x_k - x_{k-1}).$$

To verify that a continuous-time process $\tilde{S}_t$ is a continuous-time random walk with distribution p starting at the origin, it suffices to show that the paths are right-continuous with probability one, and that for all real $t_1 < t_2 < \cdots < t_k$ and $x_1, \ldots, x_k \in \mathbb{Z}^d$,

$$\mathbb{P}\{\tilde{S}_{t_1} = x_1, \ldots, \tilde{S}_{t_k} = x_k\} = \tilde{p}_{t_1}(x_1)\, \tilde{p}_{t_2 - t_1}(x_2 - x_1) \cdots \tilde{p}_{t_k - t_{k-1}}(x_k - x_{k-1}).$$

1.3 Other lattices

A *lattice* $\mathbb{L}$ is a discrete additive subgroup of $\mathbb{R}^d$. The term discrete means that there is a real neighborhood of the origin whose intersection with $\mathbb{L}$ is just the origin. While this book will focus on the lattice $\mathbb{Z}^d$, we will show in this section that this also implies results for symmetric, bounded random walks on other lattices. We start by giving a proposition that classifies all lattices.

Proposition 1.3.1 *If $\mathbb{L}$ is a lattice in $\mathbb{R}^d$, then there exists an integer $k \le d$ and elements $x_1, \ldots, x_k \in \mathbb{L}$ that are linearly independent as vectors in $\mathbb{R}^d$ such that*

$$\mathbb{L} = \{j_1\, x_1 + \cdots + j_k\, x_k, \quad j_1, \ldots, j_k \in \mathbb{Z}\}.$$

In this case we call $\mathbb{L}$ a k-dimensional lattice.

Proof Suppose first that $\mathbb{L}$ is contained in a one-dimensional subspace of $\mathbb{R}^d$. Choose $x_1 \in \mathbb{L} \setminus \{0\}$ with minimal distance from the origin. Clearly $\{jx_1 : j \in \mathbb{Z}\} \subset \mathbb{L}$. Also, if $x \in \mathbb{L}$, then $jx_1 \le x < (j+1)x_1$ for some $j \in \mathbb{Z}$, but if $x > jx_1$, then $x - jx_1$ would be closer to the origin than x_1. Hence, $\mathbb{L} = \{jx_1 : j \in \mathbb{Z}\}$.

More generally, suppose that we have chosen linearly independent $x_1, \ldots, x_j$ such that the following holds: if $\mathbb{L}_j$ is the subgroup generated by $x_1, \ldots, x_j$, and V_j is the real subspace of $\mathbb{R}^d$ generated by the vectors $x_1, \ldots, x_j$, then $\mathbb{L} \cap V_j = \mathbb{L}_j$. If $\mathbb{L} = \mathbb{L}_j$, we stop. Otherwise, let $w_0 \in \mathbb{L} \setminus \mathbb{L}_j$ and let

$$\begin{aligned} U &= \{tw_0 : t \in \mathbb{R}, tw_0 + y_0 \in \mathbb{L} \text{ for some } y_0 \in V_j\} \\ &= \{tw_0 : t \in \mathbb{R}, tw_0 + t_1x_1 + \cdots + t_jx_j \in \mathbb{L} \text{ for some } t_1, \ldots, t_j \in [0, 1]\}. \end{aligned}$$

The second equality uses the fact that $\mathbb{L}$ is a subgroup. Using the first description, we can see that U is a subgroup of $\mathbb{R}^d$ (although not necessarily contained in $\mathbb{L}$). We claim that the second description shows that there is a neighborhood of the origin whose intersection with U is exactly the origin. Indeed, the intersection of $\mathbb{L}$ with every bounded subset of $\mathbb{R}^d$ is finite (why?), and hence there are only a finite number of lattice points of the form

$$tw_0 + t_1x_1 + \cdots + t_jx_j$$

with $0 < t \le 1$; and $0 \le t_1, \ldots, t_j \le 1$. Hence, there is an $\epsilon > 0$ such that there are no such lattice points with $0 < |t| \le \epsilon$. Therefore, U is a one-dimensional lattice, and hence there is a $w \in U$ such that $U = \{kw : k \in \mathbb{Z}\}$. By definition, there exists a $y_1 \in V_j$ (not unique, but we just choose one) such that $x_{j+1} := w + y_1 \in \mathbb{L}$. Let $\mathbb{L}_{j+1}, V_{j+1}$ be as above using $x_1, \ldots, x_j, x_{j+1}$. Note that V_{j+1} is also the real subspace generated by $\{x_1, \ldots, x_j, w_0\}$. We claim that $\mathbb{L} \cap V_{j+1} = \mathbb{L}_{j+1}$. Indeed, suppose that $z \in \mathbb{L} \cap V_{j+1}$, and write $z = s_0w_0 + y_2$ where $y_2 \in V_j$. Then $s_0w_0 \in U$, and hence, $s_0w_0 = lw$ for some integer l. Hence, we can write $z = lx_{j+1} + y_3$ with $y_3 = y_2 - ly_1 \in V_j$. But, $z - lx_{j+1} \in V_j \cap \mathbb{L} = \mathbb{L}_j$. Hence, $z \in \mathbb{L}_{j+1}$. □

♣ The proof above seems a little complicated. At first glance it seems that one might be able to simplify the argument as follows. Using the notation in the proof, we start by choosing x_1 to be a nonzero point in $\mathbb{L}$ at minimal distance from the origin, and then inductively to choose x_{j+1} to be a nonzero point in $\mathbb{L} \setminus \mathbb{L}_j$ at minimal distance from the origin. This selection method produces linearly independent $x_1, \ldots, x_k$; however, it is not always the case that

$$\mathbb{L} = \{j_1 x_1 + \cdots + j_k x_k : j_1, \ldots, j_k \in \mathbb{Z}\}.$$

As an example, suppose that $\mathbb{L}$ is the five-dimensional lattice generated by

$$2\mathbf{e}_1, \;\; 2\mathbf{e}_2, \;\; 2\mathbf{e}_3, \;\; 2\mathbf{e}_4, \;\; \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_5.$$

Note that $2\mathbf{e}_5 \in \mathbb{L}$ and the only nonzero points in $\mathbb{L}$ that are within distance two of the origin are $\pm 2\mathbf{e}_j, j = 1, \ldots, 5$. Therefore, this selection method would choose (in some order) $\pm 2\mathbf{e}_1, \ldots, \pm 2\mathbf{e}_5$. But, $\mathbf{e}_1 + \cdots + \mathbf{e}_5$ is not in the subgroup generated by these points.

It follows from the proposition that if $k \leq d$ and $\mathbb{L}$ is a k-dimensional lattice in $\mathbb{R}^d$, then we can find a linear transformation $A : \mathbb{R}^d \to \mathbb{R}^k$ that is an isomorphism of $\mathbb{L}$ onto $\mathbb{Z}^k$. Indeed, we define A by $A(x_j) = \mathbf{e}_j$ where $x_1, \ldots, x_k$ is a basis for $\mathbb{L}$ as in the proposition. If S_n is a bounded, symmetric, irreducible random walk taking values in $\mathbb{L}$, then $S_n^* := AS_n$ is a random walk with increment distribution $p \in \mathcal{P}_k$. Hence, results about walks on $\mathbb{Z}^k$ immediately translate to results about walks on $\mathbb{L}$. If $\mathbb{L}$ is a k-dimensional lattice in $\mathbb{R}^d$ and A is the corresponding transformation, we will call $|\det A|$ the *density* of the lattice. The term comes from the fact that as $r \to \infty$, the cardinality of the intersection of the lattice and ball of radius r in $\mathbb{R}^d$ is asymptotically equal to $|\det A|\, r^k$ times the volume of the unit ball in $\mathbb{R}^k$. In particular, if $j_1, \ldots, j_k$ are positive integers, then $(j_1\mathbb{Z}) \times \cdots \times (j_k\mathbb{Z})$ has density $(j_1, \ldots, j_k)^{-1}$.

Examples

- The *triangular lattice*, considered as a subset of $\mathbb{C} = \mathbb{R}^2$ is the lattice generated by 1 and $e^{i\pi/3}$,

$$\mathbb{L}_{\mathrm{T}} = \{k_1 + k_2\, e^{i\pi/3} : k_1, k_2 \in \mathbb{Z}\}$$

Note that $e^{2i\pi/3} = e^{i\pi/3} - 1 \in \mathbb{L}_{\mathrm{T}}$. The triangular lattice is also considered as a graph with the above vertices and with edges connecting points that are Euclidean distance one apart. In this case, the origin has six nearest neighbors, the six sixth roots of unity. Simple random walk on the triangular lattice is

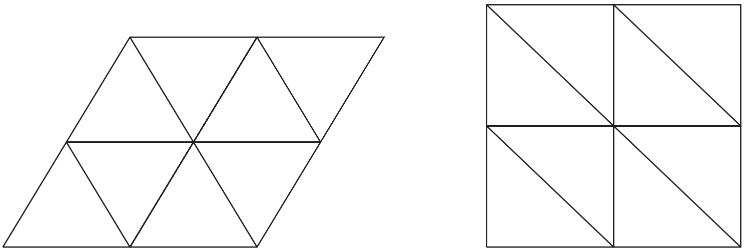

Figure 1.2 The triangular lattice $\mathbb{L}_T$ and its transformation $A\mathbb{L}_T$

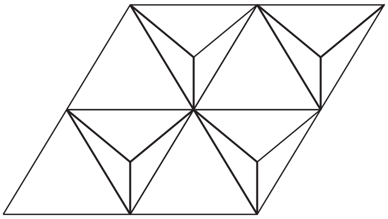

Figure 1.3 The hexagons within $\mathbb{L}_T$

the process that chooses among these six nearest neighbors equally likely. Note that this is a symmetric walk with bounded increments. The matrix

$$A = \begin{bmatrix} 1 & -1/\sqrt{3} \\ 0 & 2/\sqrt{3} \end{bmatrix}$$

maps $\mathbb{L}_T$ to $\mathbb{Z}^2$ sending $\{1, e^{i\pi/3}, e^{2i\pi/3}\}$ to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_1\}$. The transformed random walk gives probability 1/6 to the following vectors: $\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm(\mathbf{e}_2 - \mathbf{e}_1)$. Note that our transformed walk has lost some of the symmetry of the original walk.

- The *hexagonal or honeycomb lattice* is not a lattice in our sense but rather, a dual graph to the triangular lattice. It can be constructed in a number of ways. One way is to start with the triangular lattice $\mathbb{L}_T$. The lattice partitions the plane into triangular regions, of which some point up and some point down. We add a vertex in the center of each triangle pointing down. The edges of this graph are the line segments from the center points to the vertices of these triangles (see Fig. 1.3).

  Simple random walk on this graph is the process that at each time step moves to one of the three nearest neighbors. This is not a random walk in our strict sense because the increment distribution depends on whether the