1 The Problem of Spacecraft Trajectory Optimization

Bruce A. Conway
Dept. of Aerospace Engineering, University of Illinois at Urbana-Champaign, Urbana, IL

1.1 Introduction

The subject of spacecraft trajectory optimization has a long and interesting history. The problem can be simply stated as the determination of a trajectory for a spacecraft that satisfies specified initial and terminal conditions, that is conducting a required mission, while minimizing some quantity of importance. The most common objective is to minimize the propellant required or equivalently to maximize the fraction of the spacecraft that is not devoted to propellant. Of course, as is common in the optimization of continuous dynamical systems, it is usually necessary to provide some practical upper bound for the final time or the optimizer will trade time for propellant. There are also spacecraft trajectory problems where minimizing flight time is the important thing, or problems, for example those using continuous thrust, where minimizing flight time and minimizing propellant use are synonymous.

Except in very special (integrable) cases, which reduce naturally to parameter optimization problems, the problem is a continuous optimization problem of an especially complicated kind. The complications include the following: (1) the dynamical system is nonlinear; (2) many practical trajectories include discontinuities in the state variables, for example, there may be instantaneous velocity changes (also known as "ΔV's") from use of rocket motors or from planetary flyby (or "gravity assist") maneuvers, there may be instantaneous changes in spacecraft mass from staging or from using the rocket motor, or there may be sudden changes due to coordinate transformations necessary as the spacecraft moves from the gravitational sphere of influence of one body to that of another; (3) the terminal conditions, initial, final or both, may not be known explicitly, for example, for an interplanetary trajectory, the positions of the departure and arrival planets depend on the terminal times, which are often optimization variables; (4) there may be time-dependent forces, for example, the perturbations from other planets during an interplanetary trajectory can only be determined after the positions of the planets are determined using an ephemeris; (5) the basic structure of the optimal trajectory may not be *a priori* specified but is instead subject to optimization. For example, the optimal number of impulses or the optimal number of planetary flybys (or even the planets to use for the flybys) may not be known. The VEEGA trajectory for Galileo [1] is an example; this was not the only feasible trajectory but was determined to be the optimal flyby sequence.

There are many types of spacecraft trajectories. Until 1998 (and the very successful Deep Space 1 mission), spacecraft were propelled only impulsively, using chemical rockets whose burn duration is so brief in comparison to the total flight time that it is reasonable to model it as instantaneous. Between impulses, the spacecraft motion, as a reasonable first approximation, can be considered Keplerian. Interplanetary cases add the possibility of planetary flyby maneuvers, which again, as a first approximation, may be modeled as nearly instantaneous velocity changes, preceded and followed by Keplerian motion. The impulsive transfer case, even including flybys, is thus a parameter optimization problem with the parameters being such quantities as the timing, magnitude, and direction of the impulsive ΔV's and the timing and altitude of gravity assist maneuvers. Of course, for extremely accurate spacecraft trajectory optimization, the resulting approximate trajectories must be reconsidered with the perturbations of other solar system bodies, the effect of solar radiation pressure, and other small but not insignificant effects included.

While the potential benefits of low-thrust electric propulsion have been known for many years, it has only been relatively recently that spacecraft missions have been flown using this technology, for example in the NEAR and Deep Space 1 missions. Electric propulsion produces very small thrust, so that typical spacecraft acceleration is on the order of 10^{-5} g, and thus thrust is used either continuously or nearly so. The continuous thrust optimal control problem is qualitatively different from the impulsive case as there are now no integrable arcs and the control itself, for example the thrust magnitude and direction, have continuous time histories that must be modeled and determined. If the electric power is provided by solar cells, the variation of power available with distance from the sun must also be taken into consideration. A qualitatively similar continuous thrust case is that of solar sail-powered spacecraft, which of course are also subject to variation in effectiveness as they move away from the sun.

While orbit transfer, for example LEO-GEO transfer, and interplanetary trajectories have been the focus of the bulk of research into spacecraft trajectory optimization, there are certainly many other applications of optimal control theory and numerical optimization to astrodynamics. Recent interesting problems include: (1) multi-vehicle navigation and maneuver optimization for cooperative vehicles, for example a fleet of small satellites in a specified formation [2]; (2) multi-vehicle noncooperative maneuver optimization, for example pursuit-evasion problems such as the interception of a maneuvering ICBM warhead by an intercepting spacecraft or missile [3]; (3) so-called "low-energy" transfer using invariant manifolds of the three-body problem, alone [4] or in combination with conventional or low-thrust propulsion [5] and; (4) trajectory optimization for a spacecraft sent to collide with a threatening Earth-approaching asteroid, with the objective of maximizing the subsequent miss distance of the asteroid at its closest approach to Earth [6] [7]. These are only a few of many examples that could be drawn from recent literature and from the programs of the principal conferences in the subject.

Necessary conditions for optimality for every one of these types of spacecraft trajectory optimization problems may be derived using the calculus of variations (COV). Unfortunately the solution of the resulting system of equations and boundary conditions is either difficult or impossible. For certain simplified but still very useful cases of either impulsive-thrust or continuous-thrust orbit transfer, the analytical necessary conditions may described using the "Primer Vector" theory of Lawden [8], as will be described briefly in Section 1.2.1 of this Chapter and then in much greater detail in Chapter 2. Analytical solutions for the optimal trajectory (i.e. solutions satisfying the necessary conditions) can be obtained in special cases, for example for very-low-thrust orbit raising [9], even in the presence of some perturbations [10]. However, the vast majority of researchers and analysts today use numerical optimization. Numerical optimization methods for continuous optimal control problems are generally divided into two types. *Indirect* solutions are those using the analytical necessary conditions from the calculus of variations [11]. This requires the addition of the costate variables (or adjoint variables or Lagrange multipliers) of the problem, equal in number to the state variables, and their governing equations. This instantly doubles the size of the dynamical system (which alone, of course, makes it more difficult to solve). *Direct* solutions, of which there are many types, transcribe the continuous optimal control problem into a parameter optimization problem [12] [13] [14]. Satisfaction of the system equations is accomplished by integrating them stepwise using either implicit or explicit (for example Runge-Kutta) rules; in either case, the effect is to generate nonlinear constraint equations that must be satisfied by the parameters, which are the discrete representations of the state and control time histories. The problem is thus converted into a nonlinear programming problem. There is a comprehensive survey paper by Betts [15] that describes direct and indirect optimization, the relation between these two approaches, and the development of these two approaches.

1.2 Solution Methods

In just the decade since the publication of Betts' survey paper [15], there has been considerable advancement of direct numerical solutions for optimal control problems. There also has been even more development and improvement, in relative terms, of a qualitatively different approach to solving such problems, one using evolutionary algorithms. The best known of these are genetic algorithms (GA) [16]. Another evolutionary algorithm, the Particle Swarm Optimizer (PSD) will be discussed in Chapter 10. The evolutionary algorithms have two principal advantages over other extant methods; they are comparatively simple and thus easy to learn to use, and they are generally more likely, in comparison to conventional optimizers, to locate global minima. In addition, there has been progress in analytical solutions such as those using *primer vector* theory [8] [17], "shape based" trajectories [18] [19], or Hamilton-Jacobi theory.

All of the solutions may be broadly categorized as being either *analytical* or *numerical*, though of course the analytical solutions (with only a few exceptions such as the Hohmann transfer) use numerical methods and the numerical solutions include some methods that explicitly use the analytical necessary conditions for optimality. In the following sections, the analytical and numerical solution methods will be defined and various examples, some historical and some very recent, will be presented for many of the methods that fall within these categories. This is not intended to be a survey and will be unapologetically incomplete, as the subject is a vast one with a large literature. Rather, the intention in this introductory chapter is to describe the problem of spacecraft trajectory optimization, categorize the solution approaches, provide a small amount of history, and describe the "state of the art" so that the work of the various book chapter authors describing their approaches to the problem will be in context.

1.2.1 Analytical Solutions

This is the original approach for space trajectory optimization, the oldest example of which (1925) is due to Hohmann's conjecture [20] regarding the optimal circular orbit to circular orbit transfer. (The proof of the optimality of the Hohmann transfer came much later [21] [22].) Most of the analytical solutions are based on the necessary conditions of the problem that come from the calculus of variations (COV). Suppose that the system equations may be written in form

$$\dot{x} = f(x,\ u,\ t) \tag{1.1}$$

where x represents an n-dimensional state (vector) and u represents the m-dimensional control (vector). The state vector is problem dependent; there are many choices available. Typically, conventional elliptic elements, equinoctial variables, or Delaunay variables are used for problems that are Keplerian or nearly-Keplerian, for example, very low-thrust orbit raising. Another common choice is spherical polar coordinates. Cartesian coordinates are typically used for three-body problems. The control u is typically a control of thrust magnitude and direction or its equivalent, for example the orientation of a solar-sail spacecraft with respect to the Sun. The problem has some initial conditions specified, that is,

$$x_i(0) \text{ given for i} = 1, 2, \ldots, \text{k with k} \leq \text{n} \tag{1.2}$$

and some terminal conditions, or functions of the terminal conditions, specified as the vector

$$\Psi\,[x(T),\ T] = 0. \tag{1.3}$$

The objective may be written in the Bolza form as

$$J = \phi\,[x(T),\ T] + \int_0^T L\,[x,\ u,\ t]\,dt \tag{1.4}$$

where ϕ is a terminal cost function while the integral expresses a cost incurred during the entire trajectory.

The first step in deriving the conditions for an extremum of (1.4) subject to the system (1.1) and the boundary conditions (1.3) is to define a system Hamiltonian

$$H = L + \lambda^T f$$

Then, in terms of H and the other quantities introduced, the necessary conditions become [11]

$$\dot{\lambda} = -\left(\frac{\partial H}{\partial x}\right)^T \text{ with boundary condition } \lambda(T) = \left[\left(\frac{\partial \phi}{\partial x}\right) + \nu^T \left(\frac{\partial \Psi}{\partial x}\right)\right]^T_{t=T} \tag{1.5}$$

$$\frac{\partial H}{\partial u} = 0. \tag{1.6}$$

The system of equations (1.1)–(1.6) constitutes a two-point-boundary-value problem (TPBVP); some boundary conditions on the states are specified at the initial time and some boundary conditions on the states and adjoints are specified at the terminal time. In addition, if the terminal time is unspecified (that is free to be optimized), as is often the case, an additional scalar equation obtains

$$\left[\frac{\partial \phi}{\partial t} + \nu^T \left(\frac{\partial \Psi}{\partial t}\right) + \left(\frac{\partial \phi}{\partial x} + \nu^T \left(\frac{\partial \Psi}{\partial x}\right)\right) f + L\right]_{t=T} = 0. \tag{1.7}$$

For all but the most elementary optimal control problems, the solution of this TPBVP is challenging and numerical solutions are required. Despite this, it is interesting that when this set of necessary conditions is applied to the optimal space trajectory problem, which is by no means elementary, several very useful observations may be made.

The system equations of motion (1.1) may be written in the form

$$\dot{\bar{x}} = \bar{f} = \begin{bmatrix} \dot{\bar{r}} \\ \dot{\bar{v}} \end{bmatrix} = \begin{bmatrix} \bar{v} \\ \bar{g}(\bar{r}) + \Gamma \hat{u} \end{bmatrix} \tag{1.8}$$

where g(r) is the gravitational acceleration, Γ is the thrust acceleration magnitude, and $\hat{u}$ is a unit vector indicating the thrust direction.

To minimize the velocity change required, one chooses the integrand in the cost function (1.4) to be L = Γ the acceleration provided by the motor; then the integral will represent the ΔV provided by the motor. The Hamiltonian then becomes

$$H = \Gamma + \bar{\lambda}_r^T \bar{v} + \bar{\lambda}_v^T \left[\bar{g}(\bar{r}) + \Gamma \hat{u}\right] = \Gamma \left[1 + \bar{\lambda}_v^T \hat{u}\right] + \bar{\lambda}_r^T \bar{v} + \bar{\lambda}_v^T \bar{g}(\bar{r}). \tag{1.9}$$

Because H is linear in u, equation (1.6) does not obtain. The optimal control is instead chosen according to Pontryagin's Minimum Principle, stating that at any time on the optimal trajectory, the control variables are chosen in order to minimize the Hamiltonian. Thus the first simple observation is that the thrust pointing unit vector is chosen to be parallel to the opposite of the adjoint (to the velocity) vector, i.e. $-\bar{\lambda}_v(t)$. Because of its physical significance to the problem, this (adjoint) vector is referred to as the *primer vector* [8]. A second simple observation is that with this choice of thrust direction, it is then optimal in this case to choose the thrust magnitude Γ at its maximum possible value if the "switching function"

$$\left[1 + \bar{\lambda}_v^T \hat{u}\right] \tag{1.10}$$

is negative and choose $\Gamma = 0$ if the switching function is positive. The adjoint vector $\bar{\lambda}_v(t)$ is governed by the system equations (1.5) with the Hamiltonian (1.9).

In addition, it is straightforward to show that if the Hamiltonian H is not explicitly time dependent, then H is a constant on the optimal trajectory. This result is not necessarily useful for *obtaining* the optimal control but can be of great use in determining, by its use *a posteriori*, the accuracy of the numerical solution of the TPBVP, that is, a good solution will have H the same, to several significant figures, when evaluated at any point on the numerical solution [14] [23].

Finally, while the necessary conditions guarantee only that the trajectory represents an extremum of the cost, by the nature of the space trajectory problem, there is clearly no upper bound to the fuel that could be consumed on a feasible trajectory (other than consuming all the fuel available). So one may be confident that a solution is a local minimum and not a local maximum.

Further results can be obtained from a description of the necessary conditions in terms of the primer vector, and these will be described in Chapter 2. It will suffice to say here that while the primer vector is defined, and has the significance with regard to optimal thrust direction found above, this is of course true only on the optimal trajectory. The improvement of a known, nonoptimal trajectory via primer vector theory was first discussed by Lion and Handelsman [17]. Jezewski and Rozendaal [24] showed under what conditions an optimal N impulse trajectory could be improved by the addition of another impulse, and where and with what direction to apply it.

Solution of the analytical necessary conditions is possible for some special cases. One useful example is the case of very-low-thrust orbit raising. With certain assumptions, it is possible to find approximate solutions of the analytical necessary conditions. Many of these are found in a survey paper of the subject by Petropoulos and Sims [25]. The most common simplifications include: assuming that the thrust direction is always tangential; assuming that the thrust pointing is always in the direction of the velocity vector; or assuming that the orbit is always circular. Surprisingly, exact solutions also exist in certain cases, including this low-thrust orbit raising, even in the presence of nonspherical Earth perturbations [10]. This will be discussed in Chapter 7. The mathematics and analysis become very involved.

The solution of the TPBVP resulting from (or constituting) the necessary conditions becomes quite difficult for other problems, particularly those with path constraints (typically on the state variables or on functions of the state variables) or constraints on total fuel available.

Many methods have been developed to solve the TPBVP numerically. The most obvious and well known is probably shooting (an archetype of shooting applied to spacecraft trajectory optimization may be found in the paper by Breakwell and Redding [26]) but there are other methods including finite-difference methods [27] [28] and collocation [12] [13] [14]. The long-recognized difficulty of the "indirect" approach to determining the optimal trajectory is that the initial costate variables of the TPBVP are unknown and further that the nonlinearity of the problem means that the vector flow is very sensitive to some or all of these initial costate variables. A further difficulty is that the costate variables lack the physical significance of the state variables so that estimating the order of magnitude or even the sign of the initial costates is very difficult. For problems with constrained arcs, another difficulty that arises is discontinuity of controls and costate variables at the junctions of constrained and unconstrained arcs. This also increases the difficulty of solving the associated TPBVP.

Another solution method that satisfies both the necessary and sufficient conditions for optimality is the method of Static/Dynamic control (SDC) of Whiffen [29] [30]. The term static refers to decision variables that are discrete, such as launch dates or planetary flyby dates, while the term dynamic refers to controls that have a continuous variation in time, such as thrust pointing angle time histories. SDC is a general nonlinear optimal control algorithm based on Bellman's principle of optimality [11]. The implementation of SDC in the program Mystic is a very capable low-thrust spacecraft trajectory optimizer.

A recent, qualitatively different approach to the determination of optimal space trajectories is that of Guibout and Scheeres [31]. In this work, the dynamical system of state and costate variables (the vector field) is solved for specified terminal conditions and final time by solving the associated Hamilton-Jacobi (H-J) equation. The solution of the H-J equation is a generating function for a canonical transformation. Once this solution is determined, the initial value of the costate vector may be found; the optimal trajectory and the optimal control may then be found by forward integration of the flow field. Scheeres et al. show an example of an optimal rendezvous in the vicinity of a nominal circular orbit [32].

1.2.2 Numerical Solutions via Discretization

Many recent methods for solving optimal control problems seek to reduce them to parameter optimization problems that can then be solved by a NLP problem solver. One principal way in which such methods are distinguished is with regard to what quantities are parameterized. In one popular method, the collocation method that will be discussed in Chapter 3, it is possible to parameterize the state variables and the costate variables (that is, to solve the TPBVP). It is also possible in collocation to parameterize only the state variables and the control variables, as will be discussed in the next section. A third possibility, yielding the smallest number of parameters for a given problem, is to parameterize only the control variables and some free terminal states, but then the system equations must be numerically integrated (as opposed to the implicit integration that occurs in collocation). This is referred to as "control parameterization" and will be discussed in Chapter 5.

Of course all of the solutions described in the previous section are obtained numerically, that is, they will employ methods such as numerical integration, solving TPBVP problems using "shooting" methods, or solving boundary value problems by converting them into nonlinear programming (NLP) problems. What is meant in this section by "numerical solution" is solutions that do not explicitly employ the analytical necessary conditions of the COV, for example, solutions that do not employ the costate (adjoint) variables of the problem or solutions that satisfy the H-J-B equation or Bellman's principle for discrete systems.

Why would one want to avoid the use of the necessary conditions, particularly when the resulting trajectory has a "guarantee" of being a local extremum (that one loses in a numerical solution) and has other benefits previously discussed, such as information about sensitivity to terminal conditions and guidance toward improving a solution by for example, adding/subtracting thrust arcs? The principal reason is the lack of robustness of the various methods for solving the Euler-Lagrange TPBVP stemming, as previously mentioned, from the nonlinearity of the problem and a lack, in the general case, of a systematic means for determining a sufficiently good approximation to the initial adjoint variables of the problem.

A variety of direct solution methods have been developed. They are best categorized by the way in which they handle the discretization of the equations of motion, which appear as function-space constraints in the original optimal control problem. A more complete survey will be presented in Chapter 3. In the last two decades, however, the most successful approach is arguably one in which the continuous problem is discretized and state and control variables are known only at discrete times. Satisfaction of the equations of motion is achieved by employing an explicit or implicit numerical integration rule that needs to be satisfied at each step; this results in a large NLP problem with a large number of nonlinear constraints. This approach was termed "direct transcription" by Canon et al. [33]. While known to mathematicians in the 1960s and 1970s, it became known in the aerospace community principally through two papers. Dickmanns and Well [34] used the collocation scheme to solve the TPBVP of the indirect method. This approach is significantly more robust than shooting methods because it eliminates the sequential nature of the shooting solution, with its forward numerical integration, in favor of a solution in which simultaneous changes in all of the discrete state and costate parameters are made in order to satisfy algebraic constraints (while minimizing the objective of course).

However, the most useful development for space trajectory optimization was the observation in 1987 by Hargraves and Paris [12] that it was not necessary to use this approach to solve the indirect TPBVP, that in fact the adjoint variables (which had been used to determine the optimal control from Pontryagin's principle) could be removed from the solution provided that discrete control variables were introduced as additional NLP parameters. This significantly improved the robustness of the method; by eliminating the adjoint variables, the problem size is reduced almost by half, and there is no longer a need to provide the NLP problem solver with an estimate of the adjoint variables, something that is always problematic. A fortunate coincidence is that at about the same time (1980s), the NLP technology required to efficiently and robustly solve large problems became available (and has been continuously improved since then) [35] [36]. The astrodynamics community swiftly embraced this method. Many optimal spacecraft trajectories have since been determined using direct methods. The direct method has also been significantly developed in the last two decades. There are now many approaches, differing primarily (for collocation methods) on how the implicit integration rules are constructed [37]. The most common approaches are to use trapezoidal [38] or Hermite-Simpson [12] integration rule constraints, or higher-degree rules from the same Gauss-Lobatto family [13] or a Gauss-pseudospectral method [39]. There also exist commercial software packages implementing direct methods for general optimal control problems, for example DIDO [40] and SOCS [38], and even solvers specifically for space and launch vehicle trajectory optimization, for example OTIS [41] and ALTOS [42] [43].

It would be accurate to say that the great majority of optimal space trajectories are now determined numerically, with methods that do not make explicit use of the analytical necessary conditions of the problem, as will be described briefly below and in detail in Chapter 2. However, that does not mean that the necessary conditions are no longer useful. On the contrary, they provide useful information that many numerical solutions naturally lack. For example, primer vector theory can provide important information on how a solution may be improved, for example by adding thrust arcs or coast arcs or by adding impulses for an impulsive trajectory. The solution of the TPBVP of the necessary conditions also provides information on the sensitivity of the solution to changes in terminal conditions and constraints.

Fortunately, without solving the TPBVP, it is possible to make use of some of these beneficial features of the solution of the necessary conditions, as will be described in Chapter 3. This occurs because of a correspondence between the final adjoint variables of the continuous TPBVP and some Kuhn-Tucker multipliers generated in (some) numerical solutions of the trajectory optimization problem [13] [14]. With these multiplier variables available, it is possible, for example, to compute the value of the system Hamiltonian over the entire trajectory time history. For many problems in which H should be a constant, this can provide a check on the accuracy of the numerical solution. Or, knowing the final adjoints and final states from, for example, a direct solution using collocation and NLP, one can integrate the E-L equations backward to the initial time. If the initial states are recovered, one can then say that the numerical solution satisfies the analytical necessary conditions and thus represents an extremal path.

1.2.3 Evolutionary Algorithms

A qualitatively different approach, recently applied to spacecraft trajectory optimization, is the use of "evolutionary" algorithms (EA). The best known of the EAs is the genetic algorithm (GA). EA's are numerical optimizers that determine, using methods similar to those found in nature, an optimal set of discrete parameters that has been used to characterize the problem solution. The EA's have two principal advantages over all of the direct and indirect solution methods previously described in this chapter: they require no initial "guess" of the solution (in fact they generate a population of initial solutions randomly), and they are more likely than other methods to locate a global minimum in the search space rather than be attracted to a local minimum.

All of the EAs require that the problem solution be capable of being described by a relatively small, in comparison to the vector of parameters of a nonlinear program, set of discrete parameters. This can be accomplished, for spacecraft trajectory optimization problems, in a number of ways:

(1) If the trajectory can naturally be described by a finite set, for example an impulsive thrust trajectory, the parameters will be such things as times, magnitudes, and directions of impulses. Between impulses the trajectories may be determined by solving Lambert's problem. In this case a small number of parameters will suffice to completely describe the solution.
(2) If the trajectory contains non-integrable arcs, for example low-thrust arcs, it is still the case that much of the trajectory can be described with a small number of parameters such as departure and arrival dates and times for the beginning and end of thrust arcs. Quantities that must be described continuously, such as thrust magnitude or pointing time history, can be parameterized using, for example, polynomial equations in time. Then the additional parameters are a small number of polynomial coefficients [44]
(3) Low-thrust arcs can also be described using "shape-based" methods [18] [19]. In this approach, a shape, which is an analytical expression for the trajectory, can be generated from a small number of parameters such that the resulting trajectory will actually be a solution of the system equations of motion. Unfortunately the thrust time history that allows this beneficial result can only be determined *a posteriori*. An EA is then used to choose the parameters defining the shape to satisfy the boundary conditions of the problem and to minimize the cost. The resulting trajectory may not be realizable, as it may require greater thrust than is available. However the trajectory may well be satisfactory as an initial guess for a more accurate method, for example a direct method such as collocation [12] [13] [14].

In the simplest form of the genetic algorithm, the set of parameters describing the solution is written as a string or sequence of numbers. Suppose that this sequence is converted to binary form; it is then similar to a chromosome but consisting only of two possible variables, a 1 or a 0. Every sequence can be "decoded" to yield a trajectory whose cost or objective value can be determined. The first step in the GA is the generation of a "population" of sequences using a random process. The great majority of these randomly generated sequences will have very large