HILBERT TRANSFORMS

The Hilbert transform arises widely in a variety of applications, including problems in aerodynamics, condensed matter physics, optics, fluids, and engineering. This work, written in an easy-to-use style, is destined to become the definitive reference on the subject. It contains a thorough discussion of all the common Hilbert transforms, mathematical techniques for evaluating them, and a detailed discussion of their application. Especially valuable features are the tabulation of analytically evaluated Hilbert transforms, and an atlas that immediately illustrates how the Hilbert transform alters a function. These will provide useful and convenient resources for researchers.

A collection of exercises is provided for the reader to test comprehension of the material in each chapter. The bibliography is an extensive collection of references to both the classical mathematical papers, and to a diverse array of applications.

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Hilbert transforms

Volume 2

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To the memory of my mother

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Volume I

Preface

My objective in this book is to present an elementary introduction to the theory of the Hilbert transform and a selection of applications where this transform is applied. The treatment is directed primarily at mathematically well prepared upper division undergraduates in physics and related sciences, as well as engineering, and first-year graduate students in these areas. Undergraduate students with a major in applied mathematics will find material of interest in this work.

I have attempted to make the treatment self-contained. To that end, I have collected a number of topics for review in Chapter 2. A reader with a good undergraduate mathematics background could possibly skip over much of this chapter. For others, it might serve as a highly condensed review of material used later in the text. The principal background mathematics assumed of the reader is a solid foundation in basic calculus, including introductory differential equations, a course in linear and abstract algebra, some exposure to operator theory basics, and an introductory knowledge of complex variables. Readers with a few deficiencies in these areas will find a number of recommendations for further reading at the end of Chapter 2. Some of the applications discussed require the reader to be familiar with basic electrodynamics.

A focus of the book is on problem solving rather than on proving theorems. Theorems are, for the most part, not stated or proved in the most general form possible. The end-notes will typically provide additional reference sources of more detailed discussions about the various theorems presented. I have not attempted to sketch the proof of every theorem stated, but for the key results connected to the Hilbert transform, at minimum an outline of the essential elements is usually presented. Consistent with the problem-solving emphasis is that all the different techniques that I know for evaluating Hilbert transforms are displayed in the book.

I take the opportunity to introduce special functions in a number of settings. I do this for two reasons. Special functions occur widely in problems of great importance in many areas of physics and engineering, and, accordingly, it is essential that students gain exposure to this important area of mathematics. Since many Hilbert transforms evaluate to special functions, it is imperative that the reader know when to stop doing algebraic manipulations. I have incorporated several mathematical topics for which few or no applications are known to the writer. The selection process was governed in part by the potential that I thought a particular area might have in problem solving, and I have done this with the full knowledge that crystal-ball reading is an art rather than a science!

The exercises are intended as a means for the reader to test his/her comprehension of the material in each chapter. The vast majority of the problems are by design routine applications of ideas discussed in the text. A small percentage of the problems are likely to be fairly challenging for an undergraduate reader, and a few problems could be labeled rather difficult. Most readers will have no trouble deciding when they have encountered an example of this latter group.

I have compiled an extensive table of Hilbert transforms of common mathematical functions. I hope this table will be useful in three ways. First, it serves as the answer key for a number of exercises that are placed throughout the text. Since many additional Hilbert transform pairs can be established by differentiation, or by appropriate multiplicative operations, etc., this table can be used to generate a great number of exercises, much to the delight of the reader. Second, I hope it will provide a useful reference source for those looking for the Hilbert transform of a particular function. Finally, for those searching for a particular Hilbert transform not present in the table, finding related transforms may give an idea on how to approach the evaluation, and give some clues as to whether a closed form expression in terms of standard functions is likely to be possible. In several sections the table includes a few specific cases followed by the general formula. This has been done to allow the reader to access the Hilbert transform of some of the simpler special cases as quickly as possible, rather than reducing a more complicated general formula.

The mini atlas of functions and the associated Hilbert transforms given in Appendix 2 is intended to provide a visual representation for a selection of Hilbert transform pairs. I hope this will be valuable for students in the applied sciences and engineering.

The reference list is rather extensive, but is not intended to be exhaustive. There are far too many published articles on Hilbert transforms to provide a complete set of references. I have attempted to give a generous number of references to applications. Many citations are given to the classical mathematical papers on the topics of the book, and for the serious student these works can be read with great profit. The Notes section at the end of each chapter gives a guide as to where to start reading for further information on topics discussed in the chapter. Elaborations and further details on the proofs of different theorems will often be located in the references cited in the end-notes.

My final task is to thank those who have helped. Logan Ausman, Dr. Matt Feldmann, Geir Helleloid, Dr. Kai-Erik Peiponen, Dr. Ignacio Porras, Dr. Jarkko Saarinen, and Corey Schuster read various chapters and made a number of useful suggestions to improve the presentation. Dr. Walter Reid and Dr. Jim Walker gave me some helpful comments on a preliminary draft of the first three chapters. Julia Boryskina and Hristina Ninova assisted with the translation of a number of technical papers. Several other students did translations and I offer a collective thanks to them.

Julia also provided assistance in the construction of the atlas of Hilbert transforms and with a number of the figures. Ali Elgindi did some numerical checking on the table of Hilbert transforms and Julia also did a few preliminary tests. Thanks are extended to Irene Pizzie for her efforts to improve the presentation.

The author would greatly appreciate if readers would bring to his attention any errors that escaped detection. The URL http://www.chem.uwec.edu/king/forward is the web address where corrections will be posted. It is the author's intention to maintain this site actively.

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Symbols

The first occurrence or a definition is indicated by a section reference or an equation number. HT is an abbreviation for the tables of Hilbert transforms given in the Appendixes (Table A.1).

$\|a\|$	sum of the components of the multi-index a; §15.5
$\arg z$	argument of a complex number; Eq. (2.69)
A_p	the A_p condition for $1 \leq p \leq \infty$; Eq. (7.377), §7.12
$b\Omega$	boundary of a bounded domain Ω; §3.1
B	Boas transform operator; §16.4
B_n	generalization of the Boas transform operator; Eq. (16.84)
$\mathbf{B}(t)$	magnetic induction; §17.9
$\mathcal{B}$	generalization of the Boas transform operator; Eq. (16.80)
$B(a, b)$	beta function (Euler's integral of the first kind); Eq. (5.112), HT-01
$BV([a, b])$	class of functions that have bounded variation on the interval $[a, b]$; §4.25
C	designation for a contour (usually closed); §2.8.1.
C	a positive (often unspecified) constant; in derivations such a constant need not be the same at each occurrence, even though the same symbol is employed.
C	SI unit for charge, the coulomb, §19.1
$\mathbb{C}$	the set of complex numbers; §2.10
C_n	symmetry operation such that rotation by $2\pi/n$ leaves the system invariant; §21.3
C^∞	infinitely differentiable function for all points of $\mathbb{R}$; §2.15.2
C_0^∞	class of functions that are infinitely differentiable with compact support; §2.15.2
C^k	class of functions that are continuously differentiable up to order k; §2.15.2

C_0^k	class of functions that are continuously differentiable up to order k and have compact support; §2.15.2.
C_p	positive constant depending on the parameter p; often not the same at each occurrence in the sequence of steps of a proof
cas	Hartley cas function; Eq. (5.59)
$\mathcal{C}f$	Cauchy transform of the function f; Eq. (3.19)
$C(z)$	Fresnel cosine integral; Eq. (14.171), HT-01
$\mathrm{chirp}(x)$	chirp function, Exercise 18.13
$\mathrm{Ci}(x)$	cosine integral; Eq. (8.78), HT-01
$\mathrm{ci}(x)$	cosine integral; HT-01
$\mathrm{cie}(\alpha, \beta)$	cosine-exponential integral; HT-01
$\mathrm{Cie}(\alpha, \beta)$	cosine-exponential integral; HT-01
$\mathrm{Cl}_2(x)$	Clausen function; HT-01
$C_n^\lambda(x)$	Gegenbauer polynomials (ultraspherical polynomials); §9.1, Eq. (11.298), HT-01
$\mathbf{D}$	electric displacement; §19.1
$\mathcal{D}$	space of all C^∞ functions with compact support; §2.15.2, §10.2
$\mathcal{D}'$	space of all distributions on $\mathcal{D}$; §10.2
$\mathcal{D}'_+$	space of distributions with support on the right of some point; §10.2
$\mathcal{D}_{L^p}$	space of test functions; §10.2
$\mathcal{D}'_{L^q}$	space of distributions; §10.2
$\mathcal{D}'_{\mathcal{L}}$	space of distributions; Eq. (17.240)
$D(x)$	Dawson's integral; Eq. (5.32)
$D_n(\theta)$	Dirichlet kernel; Eq. (6.56)
$D_n^\lambda(x)$	ultraspherical function of the second kind; Eq. (11.299)
$-e$	electronic charge
E	identity element; §2.10, Eq. (2.150)
E	energy of a signal; Eq. (18.1)
E	one-dimensional Euclidean space
E^1	one-dimensional Euclidean space; §2.11.1
E^n	n-dimensional Euclidean space; §2.11.1
E^σ	class of entire functions of exponential type; §2.8.7, §7.4
$\mathcal{E}$	space of all C^∞ functions with arbitrary support on $\mathbb{R}$; §10.2
$\mathcal{E}(t)$	envelope function; Eq. (18.76)
$\mathcal{E}'$	space of distributions having compact support; §10.2
$E_1(x)$	exponential integral; Eq. (5.98)
E_n	eigenvalues of the unperturbed Hamiltonian; §22.4, Eq. (22.57)
$E_n(z)$	exponential integral; Eq. (14.200), HT-01
$\mathbf{E}(t)$	electric field; §17.9
$\mathbf{E_i}(\omega)$	incident electric field; Eq. (20.3)
$\mathbf{E_r}(\omega)$	reflected electric field; Eq. (20.3)

$\mathbf{E_L}$	left circularly polarized electric wave; Eq. (21.16)
$\mathbf{E_R}$	right circularly polarized electric wave; Eq. (21.17)
$\mathrm{Ei}(x)$	exponential integral function; Eq. (5.101), HT-01
$\mathrm{erf}(z)$	error function; Eq. (5.27), HT-01
$\mathrm{erfc}(z)$	complementary error function; Eq. (5.141)
$\mathbf{E}_v(z)$	Weber's function; HT-01
$\mathbf{F}$	Lorentz force; Eq. (21.50).
$f()$ or f	function (at no particular specified point); §1.2
$f(x)$	function f evaluated at the point x; §1.2
f_j	oscillator strength; §19.2
$f[n]$	element of a discrete sequence; §13.2, §13.6
$\{f[n]\}$	discrete sequence; §13.6
$f_e(x)$	even function; Eq. (4.8)
$f_o(x)$	odd function; Eq. (4.9)
$f_\downarrow(c)$	limit approaching c from $c+0$; Eq. (2.22)
$f_\uparrow(c)$	limit approaching c from $c-0$; Eq. (2.23)
$\mathcal{F}f$	fourier transform of the function f; §2.6, Eq. (2.46)
$\mathcal{F}_n f$	n-dimensional Fourier transform of the function f; §15.6
$\mathcal{F}^{-1}f$	inverse Fourier transform of the function f; §2.6, Eq. (2.47)
$\mathcal{F}_c f$	Fourier cosine transform of the function f; Eq. (5.41)
$\mathcal{F}_s f$	Fourier sine transform of the function f; Eq. (5.40)
$\mathcal{F}_N$	N-point DFT operator; §13.4
$\mathcal{F}_Q$	fractional Fourier transform; §18.10, Eq. (18.147)
$\mathfrak{F}_\alpha$	discrete fractional Fourier transform; §18.13, Eq. (18.240)
$\hat{f}$	Fourier transform of the function f; §2.6
$\tilde{f}$	conjugate series of f; Eq. (6.118); alternative notation for $\mathcal{H}f$; §6.1
f'	derivative of the function f
$f^+(z)$	function f evaluated at an interior point to a contour; Eq. (3.152)
$f^-(z)$	function f evaluated at an exterior point to a contour; Eq. (3.153)
$F_n(\theta)$	Fejér kernel; Eq. (6.63)
${}_1F_1(\alpha;\beta;x)$	Kummer's confluent hypergeometric function; Eq. (5.30), HT-01
${}_2F_1(a,b;c;z)$	hypergeometric function (or Gauss' hypergeometric function); HT-01
$\boldsymbol{F}_{\mathrm{ext}}$	external force; §17.9
$\boldsymbol{F}_{\mathrm{rad}}$	radiative reaction force; §17.9
$f(\omega, 0)$	scattering amplitude at $\theta = 0$; Eq. (19.309)
$F_{\mathbf{h}}$	scattering factor; §23.4
$\mathrm{floor}[x]$	the greatest integer $\leq x$
$G(a,x)$	Hilbert transform of the Gaussian function; §4.7. The abbreviation $G(1,x) \equiv G(x)$ is employed; §9.3
$G^{(n)}_{kl_1\cdots l_2}(t_1, t_2, \ldots, t_n)$	tensor components of the nth-order response function; §22.1

$\hbar$	Planck's constant divided by 2π
H	Hilbert transform operator on $\mathbb{R}$; Eqs. (1.2) and (1.4)
$\mathcal{H}$	Hilbert transform operator for the disc; Eq. (3.202)
$\mathbf{H}$	magnetic field; Eq. (19.7)
$\mathcal{H}_\tau$	Hilbert transform operator for period 2τ; Eq. (3.286)
Hf	Hilbert transform of the function f; §1.2
$(Hf)(x)$	Hilbert transform of the function f on the real line evaluated at the point x; Eq. (1.2)
$H_e f$	Hilbert transform of the even function f on $\mathbb{R}^+$; Eq. (4.11)
$H_o f$	Hilbert transform of the odd function f on $\mathbb{R}^+$; Eq. (4.12)
$H_1 f$	one-sided Hilbert transform of the function f; Eq. (8.18)
$H_I f$	Hilbert's integral of the function f; Eq. (7.33)
$H_n f$	n-dimensional Hilbert transform of the function f; Eq. (15.26)
$\mathcal{H}_n f$	general n-dimensional Hilbert transform of the function f in E^n; Eq. (15.2)
$\mathcal{H}_{n,\varepsilon} f$	general n-dimensional truncated Hilbert transform of the function f in E^n; Eq. (15.7)
$H_{(k)} f$	Hilbert transform of the function $f(x_1, x_2, \ldots, x_k, \ldots, x_n)$ in the variable x_k; Eq. (15.36)
H^{-1}	inverse Hilbert transform operator; Eq. (4.26)
H^+	adjoint of the Hilbert transform operator; Eq. (4.194)
H_α	fractional Hilbert transform operator; Eqs. (18.209) and (18.216)
$\mathscr{H}$	Hamiltonian for an electronic system; §22.4, Eq. (22.54)
$\mathscr{H}_0$	unperturbed Hamiltonian for an electronic system; §22.4, Eq. (22.57)
$\mathscr{H}$	space of test functions; §10.14
$\mathfrak{H}$	inner product space; §2.10.1
$\mathfrak{H}$	Hilbert space; §2.10
$H_n f$	n-dimensional Hilbert transform of the function f, for $n \geq 2$; Eq. (15.26)
$H_1(f, g)(x)$	bilinear Hilbert transform; §16.5
$H_a(f, g)(x)$	bilinear singular integral operator; Eq. (16.85)
$H_n(x)$	Hermite polynomials; §9.3, Eq. (9.39), HT-01
$\mathrm{H}(x)$	Heaviside step function; Eqs. (10.54) and (18.116)
$\mathsf{H}(\omega)$	response function for a linear system; Eq. (13.1), §18.2
$\mathsf{H}_p(\omega)$	fractional Hilbert filter; §18.9, Eq. (18.142)
$H^p(D)$	Hardy space for the unit disc; §2.10.2
H^p	Hardy space for the upper half complex plane; §2.10.2
H_ν	response function at the frequency ν; Eq. (13.3)
$H_\varepsilon f$	truncated Hilbert transform; Eq. (3.3)
$H_E f$	truncated Hilbert transform; Eq. (4.507)

$H_{\mathrm{M}} f$	maximal Hilbert transform function; Eq. (7.280)
$\mathcal{H}_{\mathrm{M}} f$	maximal Hilbert transform function; Eq. (7.282)
$H_{\mathrm{S}} F$	Hilbert–Stieltjes transform of the function F; Eq. (4.551)
H_{K}	Kober's extension of the Hilbert transform operator; §16.3
H_{R_m}	Redheffer's extension of the Hilbert transform operator; §16.2
$\mathbb{H}_j$	vectorial Hilbert transform operator; Eq. (16.100)
$H_{\theta,\varepsilon}$	truncated directional Hilbert transform operator; Eq. (16.103)
H_θ	directional Hilbert transform operator; Eq. (16.104)
H_θ	helical Hilbert transform operator; Eq. (16.131)
H_{M_θ}	directional maximal Hilbert transform operator; Eq. (16.105)
H_{M_θ}	maximal helical Hilbert transform operator; Eq. (16.132)
$H_{\mathrm{M}_{n\theta}}$	double maximal helical Hilbert transform operator; Eq. (16.136)
$H_\Gamma f$	Hilbert transform of f along the curve Γ; Eq. (16.109)
$\bar{H}_\Gamma f$	modified Hilbert transform of f along the curve Γ; Eq. (16.113)
$\mathrm{H}_{\mathrm{A}} f$	Hartley transform of a function f; Eq. (5.58)
$\mathrm{H}_{\mathrm{A}}^{-1}$	inverse Hartley transform operator; Eq. (5.60)
$H_{\pm v}^{(1)}(z),\ H_{\pm v}^{(2)}(z)$	Bessel functions of the third kind (Hankel functions of the first kind and second kind, respectively); §9.9
$h_n^{(1)}(z)$	spherical Bessel functions of the third kind; Eq. (9.131) (spherical Hankel functions of the first kind)
$h_n^{(2)}(z)$	spherical Bessel functions of the third kind; Eq. (9.132) (spherical Hankel functions of the second kind)
$h_n(x)$	Hermite–Gaussian functions; Eq. (18.179)
$\mathbf{h}_k$	discrete Hermite–Gaussian vector functions; Eqs. (18.254) and (18.255)
$\mathbf{H}_v(z)$	Struve's function; Eq. (9.77), HT-01
$H_{\mathrm{D}}\{f[n]\}$	discrete Hilbert transform of the sequence $\{f[n]\}$; Eq. (13.127)
$\{H_{\mathrm{sD}} f\}(x)$	semi-discrete Hilbert transform of the sequence $\{f[]\}$; Eq. (13.133)
$\mathcal{H}_{\mathrm{D}}\{f[n]\}$	alternative definition of the discrete Hilbert transform; Eq. (13.158)
$(\mathcal{H}_{\mathrm{D}_{pq}} x)[n]$	discrete fractional Hilbert transform; Eq. (18.269)
i	imaginary unit (engineers typically use j); §2.8
I	identity operator; §4.4
I	interval; §7.9
$\lvert I \rvert$	length of an interval; §7.9
$I_n(x)$	modified Bessel function of the first kind; HT-01
$i(t)$	input (time-dependent in general) to a system; §17.1–17.2
iff	if and only if
Im	imaginary part of a complex function
inf	infimum, the greatest lower bound of a set; §2.8

Ind f	index of a function; Eq. (11.179)
J	SI unit of energy, the joule; §19.1
$J_{\pm v}(z)$	Bessel function of the first kind; §9.6, HT-01
$\mathbf{J}_v(z)$	Anger's function; §9.12, HT-01
$j_n(z)$	spherical Bessel function of the first kind; Eq. (9.115)
k	wave number; Eq. (19.87)
$\mathbf{k}$	wave vector; §20.7
$k(x, y)$	Kernel function; §1.2, Eq. (1.3)
$K(x)$	Calderón–Zygmund kernel function; §15.1
$K_n(x)$	modified Bessel function of the third kind; HT-01
$l(I)$	length of an interval I; §2.11.1
$\mathscr{L} f$	Laplace transform of the function f; Eq. (5.91)
$\mathscr{L}_2 f$	bilateral (or two-sided) Laplace transform of the function f; Eq. (5.92)
L	class of functions that are Lebesgue integrable on a given interval; §2.11.1
$L(a, b)$	class of functions that are Lebesgue integrable on the interval (a, b); 2.11.1
L^1_{loc}	class of functions that are Lebesgue integrable on every subinterval of a given interval; Eq. (4.121)
L^2	class of functions that are Lebesgue square integrable on a given interval; §2.11.1
L^p	class of functions f such that $\|f\|^p$ is Lebesgue integrable on a given interval; §2.11.1
$L^p(\mathbb{R})$	class of functions f such that $\|f\|^p$ is Lebesgue integrable on the real line; §2.11.1
l^p	§13.11
$l^p(\mathbb{Z})$	§13.11
L^∞	class of essentially bounded functions; §2.11.1
$L^p_{2\pi}$	class of periodic functions f such that $\|f\|^p$ is Lebesgue integrable on the interval $(0, 2\pi)$. $L^p_{2\tau}$ has a similar meaning for periodic functions with period 2τ.
$L^{\alpha,p}$	class of functions f such that $\|x\|^\alpha \|f(x)\|^p$ is Lebesgue integrable on a particular interval; Eq. (7.186)
$L^p(\mu)$	class of μ-measurable functions; §7.12
$L_n(x)$	Laguerre polynomials; §9.4, Eq. (9.60)
$\mathbf{L}_v(z)$	modified Struve function; HT-01
$\text{Li}_n(z)$	polylogarithm function; HT-01
$\text{Li}_2(z)$	dilogarithm function; HT-01
Lip m	Lipschitz condition of order m; §2.3
log	logarithm to the base e; the alternative notation ln is also common usage
$\log^+ f$	maximum of $\{\log\|f\|, 0\}$; Eq. (7.74)

$\mathbf{M}$	magnetization; Eq. (20.111)
$\operatorname{mod} z$	modulus of a complex number; Eq. (2.68)
$m(E)$	measure of the set E; §2.11.1
Mf	Hardy–Littlewood maximal function; §7.9
$\mathcal{M}f$	Mellin transform of f; Eq. (5.102)
$\mathcal{M}^{-1}$	inverse Mellin transform operator; Eq. (5.107)
m	SI unit for length, the meter; §19.1
$m\{g(\lambda)\}$	distribution function of g; §4.25, §7.2, Eqs. (4.556), (7.55)
$m_{X,Y}(\omega)$	relative multiplier connecting $X(\omega)$ and $Y(\omega)$; Eq. (18.60)
$\mathbb{N}$	set of positive integers; $1, 2, 3, \ldots$
N	complex refractive index; Eq. (19.90)
N^{NL}	nonlinear complex refractive index; §22.13
$\mathcal{N}$	number of molecules per unit volume; §19.2
$n(\omega)$	angular frequency-dependent refractive index; Eq. (19.91)
$n^{\mathrm{NL}}(\omega, E)$	nonlinear refractive index; §22.13, Eq. (22.238)
$N_{\pm}(\omega)$	complex refractive indices for circularly polarized modes; §21.3, Eq. (21.47)
$n_{\pm}(\omega)$	real parts of $N_{\pm}(\omega)$; Eqs. (21.79) and (21.80)
$\mathcal{O}$	linear operator on a vector space; §2.10
$\mathcal{O}^{+}$	*adjoint* operator to $\mathcal{O}$; §2.10
$\mathcal{O}^{-1}$	inverse of an operator $\mathcal{O}$; §2.10
$O(\,)$	Bachmann order notation, of the order of; Eq. (2.1)
$o(\,)$	Landau order notation, of the order of; Eq. (2.6)
$\mathcal{O}'_C$	space of distributions that decrease rapidly at infinity; §10.2
$P\int$	Cauchy principal value; §2.4, Eq. (2.18)
$P(r, \theta)$	Poisson kernel for the disc; Eq. (3.49)
$P(x, y)$	Poisson kernel for the half plane; Eq. (3.31)
P_{ε}	Poisson operator; §7.10, Eq. (7.290)
P_{+}	projection operator; Eq. (4.352)
P_{-}	projection operator; Eq. (4.353)
$Pf(x^{-1})$	pseudofunction; §10.1
$\mathbf{P}(\mathbf{x})$	electric polarization of a medium; Eq. (19.1)
$\mathrm{P}_n(x)$	one of the orthogonal polynomials; §9.1
$P_n(x)$	Legendre polynomials; §9.2, Eqs. (9.10) and (9.27)
$P_{\nu}^{m}(x)$	associated Legendre function of the first kind; HT-01
$P_n^{(\alpha,\beta)}(x)$	Jacobi polynomials; §9.1
$\mathscr{S}_T$	space of periodic testing functions of period T; §10.2
$\mathscr{S}'_T$	space of periodic distributions of period T; §10.2
$p.v.\frac{1}{x}$	distribution; §10.1

$\mathcal{P}\frac{1}{x}$	distribution; §10.1
$Q(r,\theta)$	conjugate Poisson kernel for the disc; Eq. (3.50)
$Q(x,y)$	conjugate Poisson kernel for the half plane; Eq. (3.32)
Q_ε	conjugate Poisson operator; §7.10, Eq. (7.291)
$Q_n(x)$	Legendre function of the second kind; Eq. (11.263)
$Q_\nu^m(x)$	associated Legendre function of the second kind; HT-01
$Q_n^{(\alpha,\beta)}(x)$	Jacobi function of the second kind; HT-01
R	reflection operator; Eq. (4.73)
R	radius for a semicircular contour
A	Radon transform; §5.10, Eqs. (5.152) and (5.155)
$R_i(z_i)$	residue corresponding to the pole at $z = z_i$; §2.8.5
$R_j f$	Riesz transform of the function f; §15.12
$\mathbb{R}$	real line; the set of real numbers
$\mathbb{R}^+$	positive real axis interval; §3.4
$\mathbb{R}\times\mathbb{R}$	Euclidean plane
$\mathbb{R}^n$	n-dimensional Euclidean space; §2.15.2
$\mathcal{R}$	simply connected region; §2.8.1
$\mathcal{R}$	radius for a semicircular contour
$\mathfrak{R}_p$	Riesz constant; §4.20, Eqs. (4.382) and (4.384)
$\tilde{r}(\omega)$	generalized or complex reflectivity; Eq. (20.1)
$\tilde{r}_\pm(\omega)$	generalized reflectivity for circularly polarized modes; Eq. (21.132)
$r(\omega)$	reflectivity amplitude; Eq. (20.1)
$r(t)$	response (time-dependent) from a system; §17.1, §17.2
R_{n0}	rotational strength; Eq. (21.233)
$R(\omega)$	reflectivity; Eq. (20.2)
$re^{i\theta}$	polar form of the complex number z
Re	real part of a complex number
rect(x)	reactangular pulse function; §18.7.3
$\text{Res}\{g(z)\}_{z=z_0}$	residue at the pole $z = z_0$ of the function g; §2.8.5, Eq. (2.93)
Sf	Stieltjes transform of the function f; Eqs. (5.77), and (8.6)
S_a	dilation operator (homothetic operator); Eqs. (4.70) and (15.68)
sgn	signum function (sign function); Eqs. (1.14) and (18.120)
S^{n-1}	locus of points $x \in \mathbb{R}^n$ for which $\|x\| = 1$; §16.6
$S(z)$	Fresnel sine integral; Eq. (14.172), HT-01
$S(E)$	S-function (S-matrix); §17.12
$S(\omega)$	Fourier transform of a signal $s(t)$ in the frequency domain; §18.1, Eq. (18.2)
$s(t)$	signal in the time domain; §18.1
Shi(z)	hyperbolic sine integral function; Eq. (14.201), HT-01
Si(x)	sine integral; Eq. (8.79), HT-01

si(x)	integral; Eq. (9.170), HT-01
sie (α, β)	sine-exponential integral; HT-01
sinc x	sinc function; Eq. (4.260), HT-01
sup	supremum, the least upper bound
supp	support of the function; §2.15.2
T	finite Hilbert transform operator; chap. 11, Eq. (11.2)
T	used to denote a distribution; §2.15.2
T_{ab}	finite Hilbert transform operator on the interval (a, b); Eq. (12.98)
$T_n(x)$	Chebyshev polynomials of the first kind; §9.1, HT-01
$\mathbb{T}$	circle group; §3.10
Tr	trace; §22.4, Eq. (22.63)
$U_n(x)$	Chebyshev polynomials of the second kind; §9.1, HT-01
$u[n]$	unit step sequence; Eq. (13.91)
V	total variation of a function; Eq. (4.554)
V	SI unit for potential, the volt; §19.1
$w(x)$	weight function; §9.1
$W(x)$	weight function; §14.4
w_i	weight points in a quadrature scheme; Eq. (14.15)
$W^{p,m}$	Sobolev space; §10.2
$\bar{x}_j$	any value in the interval $[x_{j-1}, x_j]$; §2.11
x_i	sampling points in a quadrature scheme; Eq. (14.15)
$\|x\|$	norm of x in E^n; §15.1
$\mathbf{\times}$	vector cross product
$\times$	direct product; §10.6. Also used for Cartesian product of Euclidean spaces; §15.13
$\boldsymbol{x}(t)$	time-dependent particle displacement; §17.2, §17.9
$X(z)$	Z transform (one-sided or two-sided); Eqs. (13.38) and (13.39)
$Y_v(z)$	Bessel function of the second kind (Weber's function, Neumann's function); §9.6, 9.8, HT-01
$y_n(z)$	spherical Bessel function of the second kind; Eq. (9.116)
z	complex variable, $z = x + iy$; Eq. (2.67)
$\bar{z}$	complex conjugate of z
z^*	complex conjugate of z
z_I	inverse point (or image point) of z; Eq. (3.35)
$\mathbb{Z}$	set of integers $0, \pm 1, \pm 2, \ldots$
$\mathbb{Z}^+$	set of non-negative integers $0, 1, 2, \ldots$
$Z\{x_n\}$	Z transform of the sequence $\{x_n\}$; §13.6, Eq. (13.38)