15

Hilbert transforms in E^n

15.1 Definition of the Hilbert transform in E^n

In this chapter the elementary properties of the n-dimensional Hilbert transform are discussed. Basic aspects of the Calderón–Zygmund theory of singular integral operators in the n-dimensional Euclidean space, E^n, are also considered.

Applications of Hilbert transforms in E^n for $n \geq 2$ are significantly less numerous than the one-dimensional case; however, they do arise in important areas. These include problems in nonlinear optics that focus on deriving dispersion relations and sum rules for the nonlinear susceptibility. The publications of Smet and Smet (1974), Nieto-Vesperinas (1980), Peiponen, (1987b, 1988), Bassani and Scandolo (1991, 1992), and Peiponen, Vartiainen, and Asakura (1999), will give the reader a sense of the advances in this field. There are applications in signal processing (see Bose and Prabhu (1979), Zhu, Peyrin, and Goutte (1990), and Reddy *et al.* (1991a, 1991b)), and in spectroscopy (see Peiponen, Vartiainen, and Tsuboi (1990)). In scattering theory, the double dispersion relations, frequently referred to as the Mandelstam representation, express the scattering amplitude as a double iterated dispersion relation in the energy and momentum transfer variables; see Roman (1965) and Nussenzveig (1972). Some of these applications are touched upon in later chapters.

To proceed, some preliminary definitions are required. Let x denote the n-tuple $\{x_1, x_2, x_3, \ldots, x_n\}$, and let s denote the n-tuple $\{s_1, s_2, s_3, \ldots, s_n\}$. It is quite common in the literature to represent multi-dimensional integration factors by ds (or some similar variable), where the context is meant to signify an n-dimensional integration factor $ds_1 ds_2 ds_3 \cdots ds_n$. The quantity $|x|$ is defined by

$$|x| = \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2}. \qquad (15.1)$$

The sum $x + y$ is the n-tuple $\{x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\}$, and for a real constant α the quantity αx is $\{\alpha x_1, \alpha x_2, \ldots, \alpha x_n\}$. The scalar product is given by $a \cdot x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$. Common notation for the scalar product in the physical sciences is to write $\boldsymbol{a} \cdot \boldsymbol{x}$, but the use of a bold font for vector notation is postponed to later chapters discussing applications.

By analogy with what was done for the Hilbert transform on $\mathbb{R}$, the Hilbert transform in n-dimensional Euclidean space is defined by the following convolution:

$$\mathcal{H}_n f(x) = (f * K)(x) = \int_{E^n} K(x-y) f(y) \mathrm{d}y. \tag{15.2}$$

The notation T in place of $\mathcal{H}_n$ is very commonly employed. Starting in the early 1950s, Alberto Calderón and Antoni Zygmund made seminal contributions to the study of this equation, and as a result $\mathcal{H}_n$ is also referred to as the Calderón–Zygmund singular operator associated with the kernel K (see the chapter end-notes for references). The kernels are taken to be of the following form:

$$K(x) = \frac{\Omega(x')}{|x|^n}, \tag{15.3}$$

where the function Ω is defined on the unit sphere, which is denoted by Σ, and $x' = x/|x|$. Recall that a homogeneous function $h(x)$ of degree α satisfies

$$h(tx) = t^\alpha\, h(x), \quad \text{for } t > 0. \tag{15.4}$$

In the sequel it is assumed that Ω is homogeneous of degree zero, that is

$$\Omega(tx) = \Omega(x), \quad \text{for } t > 0. \tag{15.5}$$

Then Eq. (15.2) can be rewritten as follows:

$$\mathcal{H}_n f(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{\Omega(y') f(x-y) \mathrm{d}y}{|y|^n}. \tag{15.6}$$

Up to this point, the terminology Hilbert transform of f has been used to signify a one-dimensional integral on $\mathbb{R}$ or $\mathbb{T}$, so, to avoid any possibility of confusion, Eq. (15.6) will be referred to as the generalized Hilbert transform of f on E^n. The names Calderón–Zygmund transform or Calderón–Zygmund singular integral would both be better, reflecting the contributions of these outstanding mathematicians. The reader might feel that a better name is the designation *n-dimensional Hilbert transform*; however, this choice will be reserved for a particular specialization of Eq. (15.6). Equations (15.2) and (15.3), or Eq. (15.6), will be regarded as defining the so-called classical Calderón–Zygmund operators.

A truncated version of Eq. (15.2) can be defined as follows:

$$\mathcal{H}_{n,\varepsilon} f(x) = (f * K_\varepsilon)(x) = \int_{E^n} K_\varepsilon(x-y) f(y) \mathrm{d}y, \tag{15.7}$$

with the truncated kernel defined for $\varepsilon > 0$ by

$$K_\varepsilon(x) = \begin{cases} K(x), & \text{for } |x| \geq \varepsilon \\ 0, & \text{otherwise}\,. \end{cases} \tag{15.8}$$

The operator appearing in Eq. (15.7) is sometimes referred to as the truncated Calderón–Zygmund operator, and Eq. (15.8) defines a truncated Calderón–Zygmund kernel. If the limit $\varepsilon \to 0$ exists, then

$$\lim_{\varepsilon \to 0} f * K_\varepsilon = \mathcal{H}_n f. \tag{15.9}$$

Two conditions are imposed on Ω:

$$\int_\Sigma \Omega(x')\mathrm{d}x' = 0 \tag{15.10}$$

and

$$\Omega \in L^1(\Sigma). \tag{15.11}$$

The first condition is a statement that the mean value of Ω on Σ is zero. This condition is used to advantage in the sequel. The second condition is sometimes replaced by the Lipschitz condition:

$$\Omega \in \text{Lip}\,\alpha, \quad \text{for } \alpha > 0. \tag{15.12}$$

Consider first the case $n = 1$; then, the sphere Σ reduces to the two points $x' = \pm 1$, and it follows from Eq. (15.10) that $\Omega(1) + \Omega(-1) = 0$. Writing $\Omega(x') = c \operatorname{sgn} x$, with c a constant, which is assigned the value π^{-1}, yields

$$K(x) = \frac{\Omega(x')}{|x|} = \frac{\operatorname{sgn} x}{\pi |x|} = \frac{1}{\pi x}, \tag{15.13}$$

and hence Eq. (15.6) becomes, on making the identification $\mathcal{H}_1 \equiv H$,

$$Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)\mathrm{d}y}{y}. \tag{15.14}$$

This is the standard definition of the Hilbert transform on the line. Because of this reduction process, Eq. (15.6) is sometimes termed simply the Hilbert transform of f, and for this reason the notation $\mathcal{H}_n f$ has been employed. If an alternative choice is made for the constant in Eq. (15.13), the reduction process just carried out leads to a scalar multiple of the Hilbert transform.

Two results concerning the existence of $\mathcal{H}_{n,\varepsilon} f$ are now examined. The following approach is based on Neri (1971, p. 83). If $f \in L^p(E^n)$, for $1 \leq p \leq \infty$, and Ω

is bounded by a constant C, then the existence of $\mathcal{H}_{n,\varepsilon}f$ can be established in the following manner:

$$
\begin{aligned}
|\mathcal{H}_{n,\varepsilon}f(x)| &\le \int_{|y|>\varepsilon} |f(x-y)| \frac{|\Omega(y')|}{|y|^n}\,\mathrm{d}y \\
&\le C\int_{|y|>\varepsilon} \frac{|f(x-y)|}{|y|^n}\,\mathrm{d}y \\
&\le C\|f\|_p \left(\int_{|y|>\varepsilon} \frac{1}{|y|^{nq}}\,\mathrm{d}y\right)^{q^{-1}}, \qquad (15.15)
\end{aligned}
$$

where Hölder's inequality has been employed, and p and q are conjugate exponents. Both of the final integrals are finite, and hence the existence of $\mathcal{H}_{n,\varepsilon}f$ is established. The reader is requested to consider the outcome for the situations $p = 1$ and $p = \infty$. An alternative approach with weaker conditions on Ω is as follows. Let $f \in L^p(E^n)$, for $1 < p < \infty$, $\Omega \in L^1(\Sigma)$, in norm notation $\|\Omega\|_1 = \int_\Sigma |\Omega(y')|\mathrm{d}y'$, and define the integral $I(x)$ by

$$
I(x) = \int_{E^n} |f(y)||K_\varepsilon(x-y)|\mathrm{d}y = \int_{|y|>\varepsilon} |f(x-y)||K(y)|\mathrm{d}y. \qquad (15.16)
$$

If it can be shown that $I(x)$ is locally integrable, then it follows that $\mathcal{H}_{n,\varepsilon}f$ exists *a.e.*, since

$$
|\mathcal{H}_{n,\varepsilon}f(x)| \le I(x). \qquad (15.17)
$$

Let B denote any bounded set in E^n, represent $\int_B \mathrm{d}x$ by $|B|$, and let C be a constant depending on p and ε, which is not necessarily the same at each occurrence. Let the conjugate exponent of p be q, set $r = |y|$, and $y' = r^{-1}y$; then,

$$
\begin{aligned}
\int_B I(x)\mathrm{d}x &= \int_B \mathrm{d}x \int_{|y|>\varepsilon} |f(x-y)| \frac{|\Omega(y')|}{|y|^n}\,\mathrm{d}y \\
&= \int_B \mathrm{d}x \int_\Sigma |\Omega(y')|\mathrm{d}y' \int_\varepsilon^\infty \frac{|f(x-ry')|}{r}\,\mathrm{d}r \\
&= \int_\Sigma |\Omega(y')|\mathrm{d}y' \int_B \mathrm{d}x \int_\varepsilon^\infty \frac{|f(x-ry')|}{r}\,\mathrm{d}r \\
&\le C\int_\Sigma |\Omega(y')|\mathrm{d}y' \int_B \mathrm{d}x \left(\int_\varepsilon^\infty |f(x-ry')|^p\,\mathrm{d}r\right)^{p^{-1}} \\
&\le C\int_\Sigma |\Omega(y')|\mathrm{d}y' \left(\int_B \mathrm{d}x\right)^{q^{-1}} \left(\int_B \mathrm{d}x \left(\int_\varepsilon^\infty |f(x-ry')|^p\,\mathrm{d}r\right)\right)^{p^{-1}} \\
&\le C\|\Omega\|_1 |B|^{q^{-1}} \|f\|_p, \qquad (15.18)
\end{aligned}
$$

where Hölder's inequality has been applied twice. This is the required result. The case $p = 1$ is left as an exercise for the reader.

The connection between $\mathcal{H}_{n,\varepsilon} f$ and $\mathcal{H}_n f$ is now examined. Let Λ_α denote the space of Lipschitz continuous functions, with α denoting the order. Suppose $f \in L^p \cap \Lambda_\alpha$, with $1 \le p < \infty$ and $0 < \alpha \le 1$, and let Ω satisfy the conditions in Eqs. (15.10) and (15.11). Then

$$\mathcal{H}_n f = \lim_{\varepsilon \to 0} \mathcal{H}_{n,\varepsilon} f, \ a.e. \tag{15.19}$$

Select a δ such that $0 < \varepsilon < \delta$; then,

$$\mathcal{H}_{n,\varepsilon} f(x) = \int_{\delta \ge |y| \ge \varepsilon} f(x-y) \frac{\Omega(y)}{|y|^n} \mathrm{d}y + \int_{|y|>\delta} f(x-y) \frac{\Omega(y)}{|y|^n} \mathrm{d}y. \tag{15.20}$$

The first integral can be written as follows:

$$\int_{\delta \ge |y| \ge \varepsilon} |f(x-y)| \frac{\Omega(y)}{|y|^n} \mathrm{d}y = \int_{\delta \ge |y| \ge \varepsilon} [f(x-y) - f(x)] \frac{\Omega(y)}{|y|^n} \mathrm{d}y, \tag{15.21}$$

which is obtained on making use of the result

$$\int_{\delta \ge |y| \ge \varepsilon} \frac{\Omega(y)}{|y|^n} \mathrm{d}y = \int_\varepsilon^\delta \frac{\mathrm{d}r}{r} \int_\Sigma \Omega(y')\mathrm{d}y' = 0, \tag{15.22}$$

and Eq. (15.10) has been used. Since $f \in \mathrm{Lip}\,\alpha$, for $0 < \alpha$, then

$$|f(x-y) - f(x)| \le C|y|^\alpha, \quad \text{for } |y| \le \delta, \tag{15.23}$$

and the integral on the right-hand side of Eq. (15.21) is convergent in the limit $\varepsilon \to 0$, since

$$\left| [f(x-y) - f(x)] \frac{\Omega(y)}{|y|^n} \right| \le C\,|\Omega(y)||y|^{\alpha - n}, \quad \text{for } |y| \le \delta, \tag{15.24}$$

and

$$\int_{\delta \ge |y|} |\Omega(y)||y|^{\alpha-n} \mathrm{d}y = \int_0^\delta r^{\alpha-1} \mathrm{d}r \int_\Sigma \left|\Omega(y')\right| \mathrm{d}y' < \infty, \tag{15.25}$$

and Eq. (15.11) has been employed. Equation (15.19) follows from Eqs. (15.20) and (15.25). The importance of the constraint that Ω has a mean value of zero on Σ is made clear by the approach just employed.

15.2 Definition of the *n*-dimensional Hilbert transform

If the particular assignment of the constant in Eq. (15.7) is ignored, then there is only one singular integral that arises from Eq. (15.6) for the case $n = 1$. Beyond this case, there are an infinite number of choices, depending on the particular selection for Ω. One specific choice of singular integral, for $n \geq 2$, is as follows:

$$H_n f(x_1, x_2, x_3, \ldots, x_n) = \frac{1}{\pi^n} P \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \times \cdots \int_{-\infty}^{\infty} \frac{f(s_1, s_2, s_3, \ldots, s_n)\, ds_1\, ds_2\, ds_3 \ldots ds_n}{\prod_{k=1}^{n} (x_k - s_k)}. \tag{15.26}$$

The case $n = 1$ is obviously the ordinary one-dimensional Hilbert transform, and the notation H_1 is reserved for the one-sided Hilbert transform defined in Eq. (8.18). The convention adopted in Eq. (15.26) is that the Cauchy principal value applies to each integral, and the P symbol is inserted in front of each integral only when there is some risk of confusion. Using the notation introduced at the start of Section 15.1, the preceding result can be written in the more compact form:

$$H_n f(x) = \frac{1}{\pi^n} P \int_{-\infty}^{\infty} f(s) \prod_{k=1}^{n} \frac{1}{(x_k - s_k)}\, ds. \tag{15.27}$$

Equation (15.27) is frequently expressed as follows:

$$H_n f(x) = \frac{1}{\pi^n} \left\{ \prod_{j=1}^{n} \lim_{\varepsilon_j \to 0} \right\} \int_{|x_j - s_j| > \varepsilon_j} f(s) \prod_{k=1}^{n} \frac{1}{(x_k - s_k)}\, ds, \tag{15.28}$$

where the product notation for the limits implies separate limits $\varepsilon_1 \to 0, \varepsilon_2 \to 0$, and so on. As for the one-dimensional case, the opposite sign convention to that given in Eq. (15.27) is sometimes employed, that is $(s_k - x_k)$ is employed in place of $(x_k - s_k)$. Some authors do not employ the subscript notation indicating the dimensionality of the transform. Symbols other than H are also used to denote this transform, of which the most frequently employed choice is T.

Three examples of the n-dimensional Hilbert transform are now examined. Consider first the case

$$f \equiv \sin(a \cdot x) = \sin(a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_n x_n), \tag{15.29}$$

where a is a constant vector. Then it follows that

$$H_n\{\sin(a \cdot x)\} = \frac{1}{\pi^n} P \int_{-\infty}^{\infty} \frac{ds_1}{x_1 - s_1} \int_{-\infty}^{\infty} \frac{ds_2}{x_2 - s_2} \times \cdots \int_{-\infty}^{\infty} \frac{\sin(a_1 s_1 + a_2 s_2 + \cdots + a_n s_n) ds_n}{x_n - s_n}$$

$$
\begin{aligned}
&= \frac{1}{\pi^n} P \int_{-\infty}^{\infty} \frac{\mathrm{d}s_1}{x_1 - s_1} \int_{-\infty}^{\infty} \frac{\mathrm{d}s_2}{x_2 - s_2} \\
&\quad \times \cdots \int_{-\infty}^{\infty} \frac{\sin(a_1 s_1 + a_2 s_2 + \cdots + a_n x_n - a_n y)\mathrm{d}y}{y} \\
&= \frac{1}{\pi^n} P \int_{-\infty}^{\infty} \frac{\mathrm{d}s_1}{x_1 - s_1} \int_{-\infty}^{\infty} \frac{\mathrm{d}s_2}{x_2 - s_2} \cdots \\
&\quad \times \ldots \{-\cos(a_1 s_1 + a_2 s_2 + \cdots + a_n x_n)\} \int_{-\infty}^{\infty} \frac{\sin(a_n y)\mathrm{d}y}{y} \\
&= \frac{1}{\pi^n} P \int_{-\infty}^{\infty} \frac{\mathrm{d}s_1}{x_1 - s_1} \int_{-\infty}^{\infty} \frac{\mathrm{d}s_2}{x_2 - s_2} \\
&\quad \times \cdots \{-\cos(a_1 s_1 + a_2 s_2 + \cdots + a_n x_n)\} \pi \operatorname{sgn} a_n \\
&= -\frac{\operatorname{sgn} a_n}{\pi^{n-1}} P \int_{-\infty}^{\infty} \frac{\mathrm{d}s_1}{x_1 - s_1} \int_{-\infty}^{\infty} \frac{\mathrm{d}s_2}{x_2 - s_2} \cdots \\
&\quad \times \ldots \int_{-\infty}^{\infty} \frac{\cos(a_1 s_1 + a_2 s_2 + \cdots + a_{n-1} s_{n-1} + a_n x_n)\mathrm{d}s_{n-1}}{x_{n-1} - s_{n-1}},
\end{aligned}
\tag{15.30}
$$

which simplifies, on continued integration, to yield

$$
H_n \sin(a \cdot x) = \begin{cases} (-1)^{(n+1)/2} \cos(a \cdot x) \prod_{k=1}^{n} \operatorname{sgn} a_k, & \text{for } n \text{ odd} \\ (-1)^{n/2} \sin(a \cdot x) \prod_{k=1}^{n} \operatorname{sgn} a_k, & \text{for } n \text{ even.} \end{cases}
\tag{15.31}
$$

In a similar fashion,

$$
H_n \cos(a \cdot x) = \begin{cases} (-1)^{(n-1)/2} \sin(a \cdot x) \prod_{k=1}^{n} \operatorname{sgn} a_k, & \text{for } n \text{ odd} \\ (-1)^{n/2} \cos(a \cdot x) \prod_{k=1}^{n} \operatorname{sgn} a_k, & \text{for } n \text{ even.} \end{cases}
\tag{15.32}
$$

As a third example, consider

$$
f = \mathrm{e}^{-ax^2},
\tag{15.33}
$$

where $a > 0$. Then $H_n\{e^{-ax^2}\}$ is evaluated as follows:

$$\begin{aligned}
H_n\{e^{-ax^2}\} &= \frac{1}{\pi^n} P\int_{-\infty}^{\infty} \frac{ds_1}{x_1 - s_1} \int_{-\infty}^{\infty} \frac{ds_2}{x_2 - s_2} \cdots \int_{-\infty}^{\infty} \frac{e^{-a(s_1^2+s_2^2+\cdots+s_n^2)}\, ds_n}{x_n - s_n} \\
&= \frac{1}{\pi^n} P\int_{-\infty}^{\infty} \frac{e^{-as_1^2}\, ds_1}{x_1 - s_1} \int_{-\infty}^{\infty} \frac{e^{-as_2^2}\, ds_2}{x_2 - s_2} \cdots \int_{-\infty}^{\infty} \frac{e^{-as_n^2}\, ds_n}{x_n - s_n} \\
&= \{-ie^{-ax_1^2}\,\mathrm{erf}(i\surd(a)\, x_1)\}\{-ie^{-ax_2^2}\,\mathrm{erf}(i\surd(a)\, x_2)\} \\
&\quad \times \cdots \{-ie^{-ax_n^2}\,\mathrm{erf}(i\surd(a)\, x_n)\} \\
&= (-i)^n e^{-ax^2} \prod_{k=1}^{n} \mathrm{erf}(i\surd(a)\, x_k), \qquad (15.34)
\end{aligned}$$

where Eq. (5.28) has been employed.

The n-dimensional Hilbert transform operator can be written in terms of a product of one-dimensional Hilbert transform operators. The variable on which the one-dimensional operator acts is specified by a subscript thus: $H_{(k)}$. So H_n can be written as

$$H_n = \prod_{k=1}^{n} H_{(k)}, \qquad (15.35)$$

where

$$\begin{aligned}
&H_{(k)} f(s_1, s_2, \ldots, s_{k-1}, x_k, s_{k+1}, \ldots, s_n) \\
&\quad = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{f(s_1, s_2, \ldots, s_{k-1}, s_k, s_{k+1}, \ldots, s_n) ds_k}{x_k - s_k}. \qquad (15.36)
\end{aligned}$$

The operators $H_{(k)}$ satisfy the commutator condition

$$[H_{(k)}, H_{(j)}] = 0, \quad \text{for } j, k = 1, 2, \ldots, n. \qquad (15.37)$$

15.3 The double Hilbert transform

The double Hilbert transform can be used to indicate some of the issues that arise beyond the one-dimensional transform and can serve as a stepping point to higher dimensions, since the double Hilbert transform can be generalized in a rather straightforward manner. This generalization is performed in subsequent sections for a number of topics. The double Hilbert transform is defined by the following equation:

$$\begin{aligned} H_2 f(x_1, x_2) &= \frac{1}{\pi^2} P \int_{-\infty}^{\infty} P \int_{-\infty}^{\infty} \frac{f(s_1, s_2) \mathrm{d}s_1 \, \mathrm{d}s_2}{(x_1 - s_1)(x_2 - s_2)} \\ &= \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \frac{1}{\pi^2} \int_{|x_1 - s_1| > \varepsilon_1} \int_{|x_2 - s_2| > \varepsilon_2} \frac{f(s_1, s_2) \mathrm{d}s_1 \, \mathrm{d}s_2}{(x_1 - s_1)(x_2 - s_2)} \\ &= \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \int_{|x_1 - s_1| > \varepsilon_1} \int_{|x_2 - s_2| > \varepsilon_2} K(x_1 - s_1) \\ &\quad \times K(x_2 - s_2) f(s_1, s_2) \mathrm{d}s_1 \, \mathrm{d}s_2 \end{aligned} \tag{15.38}$$

where the kernel function $K(x)$ given in Eq. (15.13) is employed. With the change of variables $t_j = x_j - s_j$, for $j = 1$ or 2, the integration region is the exterior of the cross shown in Figure 15.1.

The double Hilbert transform as just defined is unique. The two-dimensional version of Eq. (15.2), namely

$$\mathcal{H}_2 f(x) = (f * K)(x) = \int_{E^2} K(x - y) f(y) \mathrm{d}y, \tag{15.39}$$

leads to an infinite number of possibilities, depending on how the kernel function K is selected. As an example, the double Hilbert transform is constrasted with a different singular integral in E^2 (discussed by Calderón and Zygmund (1952)). This case arises in potential theory. Let $f \in L$, with E^2 denoting the plane (u, v). Let (x, y, z) designate a point in the half space $z > 0$, and let R signify the distance of (x, y, z) from (u, v), that is $R^2 = (x - u)^2 + (y - v)^2 + z^2$. The potential $U(x, y, z)$ is defined by

$$U(x, y, z) = \int_{E^2} \frac{f(u, v) \mathrm{d}u \, \mathrm{d}v}{R}. \tag{15.40}$$

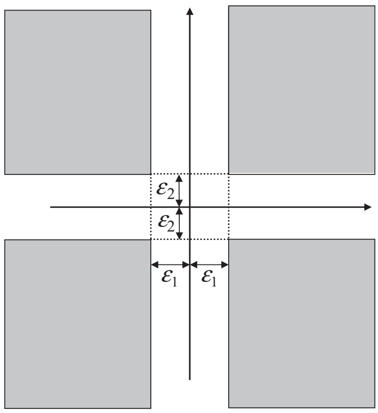

Figure 15.1. Integration domain for the double Hilbert transform.

The partial derivative of the potential, which is denoted by the appropriate subscript, is given by

$$U_x(x,y,z) = -\int_{E^2} \frac{f(u,v)(x-u)\mathrm{d}u\,\mathrm{d}v}{R^3}. \tag{15.41}$$

Set $z = 0$, then

$$U_x(x,y,0) = -\int_{E^2} f(u,v)K(x-u,y-v)\mathrm{d}u\,\mathrm{d}v, \tag{15.42}$$

where the singular kernel $K(x,y)$ is given by

$$K(x,y) = -\frac{x}{\sqrt{[(x^2+y^2)^3]}}. \tag{15.43}$$

In general, the integral in Eq. (15.42) does not converge, but it will exist *a.e.* as a principal value integral if f is a sufficiently smooth function, hence the stated requirement given for f. The principal value is interpreted here by carrying out the integral in Eq. (15.42) over the exterior of a circle of radius ε and center (x_0, y_0) and then letting $\varepsilon \to 0$.

It should be apparent to the reader from Section 15.2, and will become increasing evident in the following sections, that the factorization property of H_n allows many results about this operator to be derived relatively effortlessly from the corresponding properties for the case $n = 1$. However, there are problems where this is not the case. For example, let H_{M_n} denote the maximal Hilbert transform for the n-dimensional case. Recall Eq. (7.280) for the case $n = 1$; then, for $n = 2$,

$$H_{\mathrm{M}_2} f(x,y) = \sup_{\substack{\varepsilon_1>0,\\ \varepsilon_2>0}} \left| \frac{1}{\pi^2} \int_{|s|>\varepsilon_1} \int_{|t|>\varepsilon_2} \frac{f(x-s,y-t)}{st}\,\mathrm{d}s\,\mathrm{d}t \right|. \tag{15.44}$$

Let $f \in L\log^+ L(\mathbb{R}^2)$ and take $\operatorname{supp} f \leq [0,1]\times[0,1]$, then $(H_{\mathrm{M}_2} f)$ satisfies

$$|\{(x,y) \in [0,1]\times[0,1] : |H_{\mathrm{M}_2} f(x,y)| > \alpha\}| \leq \frac{C}{\alpha}\left\{ \left\| f \log^+ f \right\|_1 + C \right\}, \tag{15.45}$$

where the constants C and α are independent of f. Here, there is no simple factorization, and the proof of this result is relatively complicated. The interested reader can pursue this in Fefferman (1972).

15.4 Inversion property for the *n*-dimensional Hilbert transform

The results of Section 15.2 can be used to establish the inversion property for the n-dimensional Hilbert transform. Suppose

$$g(x) = H_n f(x), \tag{15.46}$$