THE CONCEPTS AND PRACTICE OF MATHEMATICAL FINANCE

Second Edition

Mathematics, Finance and Risk

Modern finance in theory and practice relies absolutely on mathematical models and analysis. It draws on and extends classical applied mathematics, stochastic and probabilistic methods, and numerical techniques to enable models of financial systems to be constructed, analysed and interpreted. This methodology underpins applications to derivatives pricing for equities and fixed income products, asset-liability modelling, volatility, risk management, credit risk, insurance analysis and many more. This new series will consist of books that explain the processes and techniques of the new applied mathematics, and how to use them to model financial systems and to understand the underlying phenomena and forces that drive financial markets.

The audience for mathematical finance ranges from mathematics and probability through econophysics to financial economics, and the series will reflect this breadth of appeal, while maintaining a firm footing in the tradition of applied mathematics. Books will be pedagogical in style, enabling them to be used for teaching in universities, business schools and financial institutions, and sufficiently self-contained for stand-alone use. Mathematical techniques will be motivated by examples and their use illustrated through applications, and complemented by computation.

THE CONCEPTS AND PRACTICE OF MATHEMATICAL FINANCE

Second Edition

M. S. JOSHI
University of Melbourne

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

www.cambridge.org
Information on this title: www.cambridge.org/9780521514088

First published 2003
Reprinted 2004
Second Edition 2008

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Joshi, M. S. (Mark Suresh), 1969–
The concepts and practice of mathematical finance / M. S. Joshi.
p. cm. – (Mathematics, finance and risk)
Includes bibliographical references and index.
ISBN 0 521 82355 2
1. Derivative securities – Prices – Mathematical models. 2. Options (Finance) – Prices – Mathematical models. 3. Interest rates – Mathematical models. 4. Finance – Mathematical models. 5. Investments – Mathematics. 6. Risk management – Mathematical models. I. Title. II. Series.
HG6024.A3J67 2003
332′.01′51 – dc22 2003055594

ISBN 978-0-521-51408-8 hardback

To My Parents

Cambridge University Press
978-0-521-51408-8 - The Concepts and Practice of Mathematical Finance, Second Edition
M. S. Joshi
Frontmatter
More information

Contents

Cambridge University Press
978-0-521-51408-8 - The Concepts and Practice of Mathematical Finance, Second Edition
M. S. Joshi
Frontmatter
More information

Preface

There are many different emphases and approaches to presenting the basics of mathematical finance. My objective in this book is to do two things: the first is to impart to the reader a conceptual understanding of the basic ideas in mathematical finance. The second is to show the reader how these ideas are translated into practicalities.

There is an aphorism that goes "Don't think of the problem, think of the solution." I believe that this aphorism is often taken too much to heart when presenting mathematical material: the solution is often presented without stating the problem. We therefore spend a couple of chapters going over the basic ideas of finance. In particular, we first introduce the concept of risk in order to give the reader an understanding of why risk is important before proving the surprising and fundamental result that ignoring risk is the key to pricing many products, which comes later in the book.

There are at least three approaches to mathematical finance, trees, PDEs and martingales. Rather than plump for one of these, we try to examine each problem from the viewpoint of each one and attempt to use the multiple approaches to emphasize the underlying ideas.

Mathematical finance is a burgeoning field and no book can cover everything, nor should it try to do so. My guiding principle has been to include what I think a good quant ought to know. Inevitably many topics are not covered in depth or at all. Where possible, I have tried to indicate other textbooks which cover the topics and where not possible the original papers. Let me stress at this point, that this is a text book not a research letter so the absence of a reference does not mean that I believe a result is new. However, on the more cutting-edge topics I have tried to indicate the original papers. If any reader is offended by the lack of a reference my apologies and please let me know for the second edition. Three books which are very strong on references are [42], [79] and [96].

After introducing risk, we move on in Chapter 2 to the concept of arbitrage which is the fundamental idea of modern derivatives pricing theory. The principle of no arbitrage is then used to develop model-free bounds on option prices, and to show that there exist certain relationships between option prices.

To pass beyond bounds to definite prices requires the introduction of a model of how asset prices change. Although a fundamental assumption is the random character of asset price movements, one must model the nature of this randomness in order to develop pricing models. In Chapter 3, we introduce the simplest of models: the binomial tree. The binomial tree is an essentially discrete model which posits that in each time period the asset moves up or down by a fixed amount. We analyze pricing on binomial trees from various points of view including replication, risk-neutral pricing and hedging. We examine the surprising result that the probabilities underlying the asset's movements have little effect on the price of options. We then see how this discrete model can be used as an approximation to a continuous model, and we deduce the Black–Scholes formula for the price of a call option via a limiting argument.

Having developed the Black–Scholes formula, we then discuss in Chapter 4 its flaws and how these flaws affect its use in practice. This chapter is very much a foretaste for chapters near the end of the book where we study alternative models of price evolution which try to compensate for the shortcomings of the Black–Scholes model.

In Chapter 5, we step up a mathematical gear and introduce the Ito calculus. With this calculus we introduce the geometric Brownian motion model of stock price evolution and deduce the Black–Scholes equation. We then show how the Black–Scholes equation can be reduced to the heat equation. This yields a derivation of the Black–Scholes formula.

In Chapter 6, we step up another mathematical gear and this is the most mathematically demanding chapter. We introduce the concept of a martingale in both continuous and discrete time, and use martingales to examine the concept of risk-neutral pricing. We commence by showing that option prices determine synthetic probabilities in the context of a single time horizon model. We then move on to study discrete pricing in martingale terms. Having motivated the definitions using the discrete case, we move on to the continuous case, and show how martingales can be used to develop arbitrage-free prices in the continuous framework. We show that the Black–Scholes PDE can be found as a consequence of the martingale method. We then move on to studying changes of numeraire and market completeness.

After the rigours of Chapter 6, we shift back to the practical in Chapter 7. In this chapter, we examine how the price of European option can be developed using the various possible pricing approaches. In particular, we discuss analytic formulas, trees, Monte Carlo, numeric integration, PDEs and replication.

In Chapter 8, we study the pricing of the simplest of exotic options, the continuous barrier option, and develop analytic formulas for its price in the Black–Scholes world using both PDE and martingale techniques. As part of the study, we examine the concept of change of measure and the reflection principle.

In Chapter 9, we commence the study of non-vanilla options by analyzing the pricing of path-dependent exotic options depending on the value of the underlying at a finite number of times. We concentrate on Asian options and discrete barrier options for concreteness. We discuss pricing using Monte Carlo and PDE methods. We also look at the computation of Greeks by Monte Carlo.

In Chapter 10, we study the use of static replication as a tool for pricing and hedging. Under a variety of assumptions, we examine the replication of continuous barrier options, discrete barrier options, and general path-dependent exotic options.

In Chapter 11, we extend the theory to cope with several sources of uncertainty and develop pricing models which can cope with derivatives whose price depends on the price behaviour of several assets. As applications of the theory, we study the pricing of Margrabe options and quanto options.

We look at how to introduce early optionality in Chapter 12. We discuss the use of tree and PDE methods before looking at the difficulties involved in pricing using Monte Carlo. We develop methods for both lower and upper bounds using Monte Carlo.

We shift our emphasis in Chapter 13 to look at the pricing of simple interest rate derivatives. We introduce forward-rate agreements and swaps, and their optional analogues the caplet and the swaption. We develop pricing formulas under simple assumptions.

In Chapter 14, we study the pricing of exotic interest rate derivatives using the LIBOR market model. Our study includes both calibration and implementation. This chapter draws on a lot of what has gone before, and we finish up with an examination of the pricing of Bermudan swaptions by Monte Carlo.

We commence our study of alternative pricing models in Chapter 15. Here we analyze the Merton jump-diffusion model and develop a pricing formula. We also discuss the additional issues raised by pricing in a model that does not allow perfect hedging.

We continue our study of alternative models in Chapter 16 where we introduce stochastic volatility. We develop pricing approaches using PDE and Monte Carlo techniques for vanilla and exotic options.

In Chapter 17, we introduce the Variance Gamma model and use it to study the pricing of vanilla and exotic options.

To round off the main part of the book, we finish with a chapter on the philosophical and practical issues inherent in using sophisticated models to price exotic options. We look at the relationship between models and smile dynamics, and compare these dynamics to those found in the market. We also see that for certain products there are features which are crucial to capture.

Preface to the Second Edition

It is now four years since the first edition appeared, and almost six since the main draft was finished. Perhaps, the biggest change during those years is the plethora of books on financial mathematics that have been published. When I commenced writing *Concepts* there was only a handful, and it was clear that there was room for a fresh approach which motivated me to write the book. Now every reasonable approach has been tackled at least once, and often several times. Yet *Concepts* has continued to be successful, perhaps because of its unique blend of mathematics and practicality.

Whilst the discipline of financial mathematics has advanced greatly in six years, the basics that an incomer to the field needs to know have not changed hugely. The main difference is that banks have much higher expectations of entry-level candidates. In 1999, demonstration of strong mathematics skills and the ability to derive the Black–Scholes equation was enough to get a job; now many candidates have Masters in Financial Engineering, sometimes as well as PhDs in other fields. Yet the material covered here plus programming skills is still sufficient to land that first job.

For that reason, in this edition, there has been a conscious decision not to include new topics. Instead, the emphasis has been placed on clarifying old topics, introducing extra references to new material and books, and on the exercises. In particular, following feedback from my students at the University of Melbourne, over fifty new exercises have been added and detailed solutions have been included for these. In addition, full solutions have been included for most exercises in the early chapters where previously only hints had been given.

New topics have instead been relegated to a sequel *More Mathematical Finance* which will, I hope, appear in the not too distant future. It will adopt a similar style but go into more details on advanced topics.

The web site for this book continues to be

`www.markjoshi.com/concepts`

There is now a bulletin board there: I encourage you to visit this and ask questions about mathematical finance as explained in this and other books.

Mark Joshi
Melbourne, January 2008

Acknowledgements

Writing this book has been a project of vastly greater magnitude than I contemplated when I started out with the objective of writing a book that stressed the ideas of mathematical finance more than the mathematics. I am grateful to the Royal Bank of Scotland for providing a stimulating environment in which to learn, study and do mathematical finance. My views on and understanding of the topic have come from daily discussions with current and former members of the Group Risk Management Quantitative Research Centre including Chris Hunter, Peter Jäckel, Dherminder Kainth, Jan Kwiatkowski and Jochen Theis, and particularly Riccardo Rebonato. I am also grateful to numerous people for their many comments on the manuscript, particularly to Alex Barnard, Dherminder Kainth, Alan Lewis, Sukhdeep Mahal, Riccardo Rebonato and Jochen Theis. David Tranah, my editor at Cambridge University Press, has done a careful job of editing and has succeeded in removing the worst quirks in my style. My wife has been very supportive during a project that at times seemed neverending.