Numerical Relativity
Solving Einstein's Equations on the Computer

Aimed at students and researchers entering the field, this pedagogical introduction to numerical relativity will also interest scientists seeking a broad survey of its challenges and achievements. Assuming only a basic knowledge of classical general relativity, this textbook develops the mathematical formalism from first principles, then highlights some of the pioneering simulations involving black holes and neutron stars, gravitational collapse and gravitational waves.

The book contains 300 exercises to help readers master new material as it is presented. Numerous illustrations, many in color, assist in visualizing new geometric concepts and highlighting the results of computer simulations. Summary boxes encapsulate some of the most important results for quick reference. Applications covered include calculations of coalescing binary black holes and binary neutron stars, rotating stars, colliding star clusters, gravitational and magnetorotational collapse, critical phenomena, the generation of gravitational waves, and other topics of current physical and astrophysical significance.

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Preface

What is numerical relativity?

General relativity – Einstein's theory of relativistic gravitation – is the cornerstone of modern cosmology, the physics of neutron stars and black holes, the generation of gravitational radiation, and countless other cosmic phenomena in which strong-field gravitation plays a dominant role. Yet the theory remains largely untested, except in the weak-field, slow-velocity regime. Moreover, solutions to Einstein's equations, except for a few idealized cases characterized by high degrees of symmetry, have not been obtained as yet for many of the important dynamical scenarios thought to occur in nature. With the advent of supercomputers, it is now possible to tackle these complicated equations numerically and explore these scenarios in detail. That is the main goal of numerical relativity, the art and science of developing computer algorithms to solve Einstein's equations for astrophysically realistic, high-velocity, strong-field systems.

Numerical relativity has become one of the most powerful probes of relativistic spacetimes. It is the tool that allows us to recreate cataclysmic cosmic phenomena that are otherwise inaccessible in the conventional laboratory – like gravitational collapse to black holes and neutron stars, the inspiral and coalescence of binary black holes and neutron stars, and the generation and propagation of gravitational waves, to name a few. Numerical relativity picks up where post-Newtonian theory and general relativistic perturbation theory leave off. It enables us to follow the full nonlinear growth of relativistic instabilities and determine the final fate of unstable systems. Numerical relativity can also be used to address fundamental properties of general relativity, like critical behavior and cosmic censorship, where analytic methods alone are not adequate. In fact, critical behavior in gravitational collapse is an example of a previously unknown phenomenon that was first discovered in numerical experiments, triggering a large number of analytical studies.

Building a numerical spacetime on the computer means solving equations. The equations that arise in numerical relativity are typically multidimensional, nonlinear, coupled partial differential equations in space and time. They have in common with other areas of computational physics, like fluid dynamics, magnetohydrodynamics, and aerodynamics, all of the usual problems associated with solving such nontrivial systems of equations. However, solving Einstein's equations poses some additional complications that are unique to general relativity. The first complication concerns the choice of *coordinates*. In general relativity, coordinates are merely labels that distinguish points in spacetime; by themselves coordinate intervals have no physical significance. To use coordinate intervals to determine physically measurable proper distances and proper times requires the spacetime metric, but the metric is known only *after* Einstein's equations have been solved. Moreover, as the numerical integrations that determine the metric proceed, the original, arbitrary choice of coordinates often turns out to be bad, because, for example, *singularities* appear in the equations. Encountering such singularities, be they physical or coordinate, results in some of the terms in Einstein's equations becoming infinite, potentially causing overflows in the computer output and premature termination of the numerical integration. It is not always easy to exploit successfully the gauge freedom inherent in general relativity – the ability to choose coordinates in an arbitrary way – and avoid these singularities in a numerical routine.

Treating *black holes* is one of the main goals of numerical relativity, but this poses another complication. The reason is that black holes contain physical spacetime singularities – regions where the gravitational tidal field, the matter density and the spacetime curvature all become infinite. Thus, when dealing with black holes, it is crucial to choose a computational technique that avoids encountering their interior spacetime singularities in the course of the simulation.

Another complication arises in the context of one of the most pressing goals of numerical relativity – the calculation of waveforms from promising astrophysical sources of *gravitational radiation*. Accomplishing this task is necessary in order to provide theoretical waveform templates both for ground-based and space-borne laser interferometers now being designed, constructed and placed into operation world-wide. These theoretical templates are essential for the identification and physical interpretation of gravitational wave sources. However, the gravitational wave components of the spacetime metric usually constitute small fractions of the smooth background metric. Moreover, to extract the waves from the background in a simulation requires that one probe the numerical spacetime in the far-field, or radiation, zone, which is typically at large distance from the strong-field central source. Yet it is the strong-field region that usually consumes most of the computational resources (e.g., spatial resolution) to guarantee accuracy. Furthermore, waiting for the wave to propagate to the far-field region usually takes nonnegligible integration time. Overcoming these difficulties to reliably measure the wave content thus requires that a numerical scheme successfully cope with the problem of vast *dynamic range* – the presence of disparate length and time scales – inherent in a numerical relativity simulation.

These are just some of the subtleties that must be confronted when doing numerical relativity. The payoff is the ability to build a spacetime on the computer that simulates the unfolding of some of the most exciting and exotic dynamical phenomena believed to occur in the physical Universe. Generating such a spacetime – "spacetime engineering" – then allows for an intimate probing of events and physical regimes that cannot be reproduced on Earth and may even be difficult to observe with telescopes. For those that can be detected, numerical relativity is a tool that can be called upon to interpret the observed features.

About this book

The purpose of this book is to provide a basic introduction to numerical relativity for nonexperts. It is a summary of the fundamental concepts as well as a broad survey of some of its most important applications. The book was conceived and written as a guide for readers who want to acquire a working knowledge of the subject, so that on mastery of the material, they can read and critique the scientific literature and begin active research in the field. Our book was born out of necessity: we needed a comprehensive guide to train our own students who want to pursue research with us in numerical relativity. Since we were unable to identify a suitable text to provide such an overview, we decided to write a book ourselves and fill the void.[1] As constructed, the book should also serve as a useful reference for researchers in the field of numerical relativity, as well as a primer for scientists in other areas desiring to get acquainted with our discipline and some of its most significant achievements.

Readers of our book are assumed to have a solid background in the basic theory of general relativity. There are several excellent textbooks that provide such a background. We are most familiar with *Gravitation* by C. W. Misner, K. S. Thorne and J. A. Wheeler (MTW) and will occasionally refer readers to this book for background material. We assume that our readers already have mathematical familiarity with tensors and differential geometry at the level of MTW, or a comparable graduate-level textbook on general relativity, and that they already have surveyed most of the physical applications covered in that book. This prerequisite roughly translates into a basic understanding of the geometric concepts and objects that enter the Einstein field equations, as well as the equations of motion for geodesics and relativistic fluids, the equations of hydrostatic equilibrium for spherical relativistic stars, the geometric and physical properties of black holes, the nature of gravitational radiation, and the concept of gravitational collapse. Beyond these standard topics, which we briefly review in Chapter **??**, our book is essentially self-contained.

The question arises as to whether readers either with little or no acquaintance with general relativity can learn something about numerical relativity by reading this book. The question might be especially relevant for experts in other disciplines with related skills, such as computational physicists and astrophysicists, computer scientists, or mathematicians. The answer is that we don't know the answer, but we are eager to find out! It is a fact that when expressed in numerical terms, many of the equations arising in numerical relativity have a form similar to equations found in many other computational disciplines (e.g., fluid dynamics). It is also a fact that advances in the field of numerical relativity have benefited enormously from developments in other fields of computational physics and computer science. We thus hope that colleagues in these and other areas continue to venture into

[1] Apparently we were not alone in recognizing this void; well into our own writing another book on numerical relativity appeared, *Introduction to 3+1 Numerical Relativity*, by **?**.

numerical relativity, and we look forward to learning from them to what extent our book can be of assistance.

To be useful as a textbook, our book contains 300 exercises scattered throughout the text. These exercises vary in scope and difficulty. They are included to assist students and instructors alike in calibrating the degree to which the material has been assimilated. The exercises comprise integral components of the main discussion in the book, so that is why they are inserted throughout the main body of text and not at the end of each chapter. The results of the exercises, and the equations derived therein, are often referred to in the book. We thus urge even casual readers who may not be interested in working through the exercises to peruse the problems and to make a mental note of what is being proven.

The book is designed as a general survey and a practical guide for learning how to use numerical relativity as a powerful tool for tackling diverse physical and astrophysical applications. Not surprisingly, the flavor of the book reflects our own backgrounds and interests. The mathematical presentation is not formal, but it is sound. We believe our overall approach is adequate for the main task of training students who seek to work in the field.

The organization of the book follows a systematic development. We begin in Chapter 1 with a *very* brief review (more of a reminder) of some elementary results in general relativity. In Chapter 2 we recast the equations of general relativity into a form suitable for solving an initial value problem in general relativity, i.e., a problem whereby we determine the future evolution of a spacetime, given a set of well-posed initial conditions at some initial instant of time. Specifically, we recast the familiar covariant, 4-dimensional form of the Einstein gravitational field equations into the equivalent $3+1$-dimensional Arnowitt–Deser–Misner (ADM) set of equations. This ADM decomposition effectively slices 4-dimensional spacetime into a continuous stack of 3-dimensional, space-like hypersurfaces that pile up along a 1-dimensional time axis. Two distinct types of equations emerge for the gravitational field in the course of this decomposition: "constraint" equations, which specify the field on a given spatial hypersurface (or "time slice"), and "evolution" equations, which describe how the field changes in time in advancing from one time slice to the next. In Chapter 3 we discuss approaches for solving the constraint equations for the construction of suitable initial data, and we provide some simple examples. In Chapter 4 we summarize a few different coordinate choices (gauge conditions) that have proven useful in numerical evolution calculations. Chapter 5 deals with the right-hand side of Einstein's equations, cataloging some different relativistic stress-energy sources that arise in realistic astrophysical applications, together with their equations of motion. Hydrodynamic and magnetohydrodynamic fluids, collisionless gases, electromagnetic radiation, and scalar fields are all represented here.

This is not a book on numerical methods *per se*. Rather, our emphasis is on deriving and interpreting geometrically various formulations of Einstein's equations that have proven useful for numerical implementation and then illustrating their utility by showing results of numerical simulations that employ them. We do not, for example, present finite difference or other discrete forms of the continuum equations, nor do we provide numerical code. But in Chapter 6 we do review some of the basic numerical techniques used to integrate standard elliptic, hyperbolic and parabolic partial differential equations, and we discuss some methods that help calibrate the accuracy of numerical solutions. These basic techniques comprise the building blocks on which all numerical implementations of the continuum formulations of Einstein's equations are based.

No object is more central to numerical relativity than the black hole. Black holes are featured throughout the book. Chapter 7 discusses some of the quantities (i.e., horizons) that help us locate and diagnose the properties of black holes residing in a numerical spacetime.

As we turn toward physical applications, our discussion proceeds in order of decreasing spacetime symmetry and increasing computational challenge. Some of the spacetimes we build involve vacuum black holes, others contain relativistic matter in various forms. Many of the examples address topical issues in relativistic gravitation or relativistic astrophysics. A substantial fraction are drawn from our own work, a choice triggered by our familiarity with this material and its accessibility, including illustrations. We hope that our colleagues will understand, and forgive us, if we seem to have overrepresented our own work as a result of this choice.

Chapter 8 constructs numerical spacetimes in spherical symmetry, which provides useful insight into gravitational collapse and black hole formation with minimal resources, but is devoid of gravitational waves. To treat gravitational waves we need to abandon spherical symmetry (Birkhoff's theorem!). To set the stage, Chapter 9 reviews some of the basic properties, plausible astrophysical sources, and current and future detectors of gravitational waves, as well as standard extraction techniques for gravitational waves in numerical spacetimes. Chapter 10 then begins our discussion of nonspherical, radiating spacetimes by featuring the collapse of collisionless clusters in axisymmetry.

To maintain long-term numerical stability during simulations in $3 + 1$ dimensions, it proves necessary to modify the ADM system of equations. Chapter 11 shows why this is true and provides alternative formulations in common use that are stable and robust.

Chapters 12 and 13 focus on the inspiral and coalescence of binary black holes, one of the most important applications of numerical relativity and a promising source of detectable gravitational radiation. These chapters treat the two-body problem in classical general relativity theory, and its solution represents one of the major triumphs of numerical relativity. Chapter 12 generates initial data for two black holes in quasistationary circular orbit, the astrophysically most realistic prelude to coalescence. Chapter 13 discusses dynamical simulations of the plunge, merger and ringdown of the two black holes and the associated waveforms. Chapter 14 treats rotating relativistic fluid stars, including numerical equilibrium models and simulations dealing with secular and dynamical instabilities and catastrophic collapse to black holes and neutron stars. Chapters 15 and 16 are the analogs of Chapters 12 and 13 for binary neutron stars. The inspiral and merger of binary neutron stars is not only a promising source of gravitational waves, but also a plausible candidate for at least one class of gamma-ray burst sources. So are black hole-neutron star binaries, which we take up in Chapter 17.

Our book could not have been written without the encouragement and insights provided by our colleagues and collaborators in numerical relativity and related areas. The individuals whose expertise we have drawn on over the years are far too numerous to list here, but we would be totally remiss if we did not thank G. B. Cook, M. W. Choptuik, C. F. Gammie, T. Nakamura, F. A. Rasio, M. Shibata, L. L. Smarr, S. A. Teukolsky, K. S. Thorne, and J. W. York, Jr. for their mentoring. We are very grateful to A. M. Abrahams, M. D. Duez, Y. T. Liu and H. J. Yo for furnishing invaluable notes and to our research groups for material that has found its way into this volume. We thank A. R. Lewis, R. Z. Gabry and A. H. Currier for helping us generate the 3-dimensional geometric illustrations in our book, to P. Spyridis for producing several line plots, and to Z. B. Etienne for providing indispensable technical assistance throughout the writing process. This project would not have been initiated without the support of G. A. Baym, D. K. Campbell, F. K. Lamb, F. K. Y. Lo, B. G. Schmidt and P. R. Shapiro, to whom we are indebted. We gratefully acknowledge the National Science Foundation, the National Aeronautics and Space Administration, and the John Simon Guggenheim Memorial Foundation for funding our research. Finally, we thank our families, to whom we dedicate this volume, for their devotion, encouragement and patience.

As the numerical algorithms continue to be refined and incorporate more physics, and as computer technology continues to advance, we anticipate that numerical relativity will accelerate in importance and use in the future. We can already foreshadow the day when youngsters are routinely downloading simulations of black hole binary coalescence on their iPods, or playing video games involving colliding neutron stars on their video cell phones, or on some new device that we cannot yet imagine! It is our fervent hope that some of the more curious will be motivated to dial into our book and learn something about the physics and mathematics underlying these remarkable simulations, so that they, in turn, may be inspired to produce the next generation of simulations that can go further toward unraveling the mysteries of nature.

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Suggestions for using this book

Our book is intended both as a general reference for researchers and as a textbook for use in a formal course that treats numerical relativity. We envision that there are at least two different ways in which the book can be used in the classroom: as the main text for a one-semester course on numerical relativity for students who have already taken an introductory course in general relativity, or as supplementary reading in numerical relativity at the end of an introductory course in general relativity. There *may* be more material in the book than can be covered comfortably in a single semester devoted entirely to numerical relativity. There *certainly* is more material than can be integrated into a supplementary unit on numerical relativity in an introductory course on general relativity. The latter may be true even when such a course is taught as a two-semester sequence, if the course is already broad and comprehensive without numerical relativity.

There are several ways to design a shortened presentation of the material in our book without sacrificing the core concepts or interfering with the logical flow. The amount of material that must be cut out from any course depends, naturally, on the amount of time that is available to devote to the subject. One means of reducing the content while retaining the fundamental ideas in a self-contained format is to restrict the discussion to pure *vacuum* spacetimes, i.e., spacetimes with no matter sources. Such spacetimes can contain gravitational waves and black holes, including binary black holes, but nothing else. Since the solution of the binary black hole problem in vacuum constitutes one of the main triumphs of numerical relativity, and since binary black hole inspiral and merger constitutes one of the most promising sources of detectable gravitational waves, one can still explore a seminal and timely topic in its entirety, even with the restriction to vacuum spacetimes. Of course, all astrophysical applications involving either hydrodynamic or magnetohydrodynamic matter, collisionless matter, or scalar fields, and whole classes of relativistic objects, like neutron stars, supernovae, collapsars, supermassive stars, collisionless clusters, etc. must then be omitted.

We provide a "roadmap" through our book in Table 1 for instructors who wish to restrict their discussion to vacuum spacetimes. The chapters and sections earmarked for inclusion constitute a respectable and self-contained "minicourse" on numerical relativity. Pointers to the relevant appendices are found in these chapters at the appropriate places. In all the sections designated in the table, all matter source terms that are retained in the gravitational field equations can be set to zero. Instructors who have time to cover more ground, but not the entire book, can then augment their discussion by adding material in the book

Table 1 Vacuum spacetime "minicourse".

Chapter	Sections
??	**??**, **??**
??	all
??	all
??	all
??	omit
??	all
??	all
??	**??**
??	all, but black holes only in **??**
??	omit
??	all
??	all
??	all
??	omit
??	omit
??	omit
??	omit

involving matter sources on a selective basis. For example, scalar field collapse and critical phenomena are developed in Chapters **??** and **??**. Collisionless matter evolution and cluster collapse and collisions are discussed in Chapters **??**, **??**, **??**, **??**, and **??**. Hydrodynamic and magnetohydrodynamic matter evolution, stellar collapse and stellar collisions are treated in Chapters **??**, **??**, **??**, **??**, **??**, and **??**–**??**. Each of these topics is developed independently of the others in the book, to first approximation, but they do rely on material covered in earlier chapters of the "minicourse".

There are, of course, other ways to parse and select from the material in the book to fit into a given course schedule. We shall leave it to individual instructors to arrange an alternative program that best suits their aims and the needs of their students.