Local Cohomology

This Second Edition of a successful graduate text provides a careful and detailed algebraic introduction to Grothendieck's local cohomology theory, including in multi-graded situations, and provides many illustrations of the theory in commutative algebra and in the geometry of quasi-affine and quasi-projective varieties. Topics covered include Serre's Affineness Criterion, the Lichtenbaum–Hartshorne Vanishing Theorem, Grothendieck's Finiteness Theorem and Faltings' Annihilator Theorem, local duality and canonical modules, the Fulton–Hansen Connectedness Theorem for projective varieties, and connections between local cohomology and both reductions of ideals and sheaf cohomology.

The book is designed for graduate students who have some experience of basic commutative algebra and homological algebra, and also for experts in commutative algebra and algebraic geometry. Over 300 exercises are interspersed among the text; these range in difficulty from routine to challenging, and hints are provided for some of the more difficult ones.

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Cambridge University Press
978-0-521-51363-0 — Local Cohomology
2nd Edition
Frontmatter
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An Algebraic Introduction with Geometric Applications

SECOND EDITION

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CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521513630

First published 1998
Second Edition 2013

Printed and bound in the United Kingdom by the MPG Books Group

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-51363-0 Hardback

To Alice

from the second author

Contents

Preface to the First Edition

One can take the view that local cohomology is an algebraic child of geometric parents. J.-P. Serre's fundamental paper 'Faisceaux algébriques cohérents' [77] represents a cornerstone of the development of cohomology as a tool in algebraic geometry: it foreshadowed many crucial ideas of modern sheaf cohomology. Serre's paper, published in 1955, also has many hints of themes which are central in local cohomology theory, and yet it was not until 1967 that the publication of R. Hartshorne's 'Local cohomology' Lecture Notes [25] (on A. Grothendieck's 1961 Harvard University seminar) confirmed the effectiveness of local cohomology as a tool in local algebra.

Since the appearance of the Grothendieck–Hartshorne notes, local cohomology has become indispensable for many mathematicians working in the theory of commutative Noetherian rings. But the Grothendieck–Hartshorne notes certainly take a geometric viewpoint at the outset: they begin with the cohomology groups of a topological space X with coefficients in an Abelian sheaf on X and supports in a locally closed subspace.

In the light of this, we feel that there is a need for an algebraic introduction to Grothendieck's local cohomology theory, and this book is intended to meet that need. Our book is designed primarily for graduate students who have some experience of basic commutative algebra and homological algebra; for definiteness, we have assumed that our readers are familiar with many of the basic sections of H. Matsumura's [50] and J. J. Rotman's [71]. Our approach is based on the fundamental 'δ-functor' techniques of homological algebra pioneered by Grothendieck, although we shall use the 'connected sequence' terminology of Rotman (see [71, pp. 212–214]).

However, we have not overlooked the geometric roots of the subject or the significance of the ideas for modern algebraic geometry. Indeed, the book presents several detailed examples designed to illustrate the geometrical significance of aspects of local cohomology; we have chosen examples which require only basic ideas from algebraic geometry. In this spirit, there is one particular example, which we refer to as 'Hartshorne's Example', to which we return several times in order to illustrate various points.

The geometric aspects are, in fact, nearer the surface of our treatment than might initially be realised, because we make much use of ideal transforms and their universal properties, but it is only in the final chapter that we expose the fundamental links, expressed by means of the Deligne Correspondence, between the ideal transform functors and their right derived functors on the one hand, and section functors of sheaves and sheaf cohomology on the other.

We define the local cohomology functors to be the right derived functors of the appropriate torsion functor, although we establish in the first chapter that one can also construct local cohomology modules as direct limits of 'Ext' modules; we also present alternative constructions of local cohomology modules, one via cohomology of Čech complexes, and the other via direct limits of homology modules of Koszul complexes, in Chapter 5. (In fact, we do not use this Koszul complex approach very much at all in this book.)

Chapters 2, 3 and 4 include fundamental ideas concerning ideal transforms and their universal properties, the Mayer–Vietoris Sequence for local cohomology and the Independence and Flat Base Change Theorems: we regard all of these as technical cornerstones of the subject, and we certainly use them over and over again.

The main purpose of Chapters 6 and 7 is the presentation of some of Grothendieck's important vanishing theorems for local cohomology, which relate such vanishing to the concepts of dimension and grade. This work is mainly 'algebraic' in nature. In Chapter 8, we present another vanishing theorem for local cohomology modules, namely the local Lichtenbaum–Hartshorne Vanishing Theorem: this has an 'analytic' flavour, in the sense that it is intimately related with 'formal' methods and techniques, that is, with passage to completions of local rings and with the structure theory for complete local rings. The Lichtenbaum–Hartshorne Theorem has important geometric applications: for example, we show in Chapter 19 how it can be used to obtain major results about the connectivity of algebraic varieties.

Grothendieck's Finiteness Theorem and G. Faltings' Annihilator Theorem for local cohomology are the main subjects of Chapter 9. These two theorems also have major geometric applications, including, for example, in Macaulayfication of schemes. They also have significance for the theory of generalized Cohen–Macaulay modules and Buchsbaum modules, two concepts which feature briefly in the exercises in Chapter 9.

We have delayed the introduction of duality (until Chapters 10 and 11) because quite a lot can be achieved without it, and because, for our discussion of duality, we have had to assume (on account of limitations of space) that the reader is familiar with the Matlis–Gabriel decomposition theory for injective modules over a commutative Noetherian ring (although we have reviewed that theory and provided some detailed proofs). We have not explicitly used dualizing complexes or derived categories, as it seems to us that such technicalities could daunt youthful readers and are not essential for a presentation of the main ideas. After the introduction of local duality in Chapter 11, we show how this duality can be used to derive some results established earlier in the book by different means.

The many recent research papers involving local cohomology of graded rings illustrate the importance of this aspect, and we have made some effort to develop the fundamentals of local cohomology in the graded case carefully in Chapters 12 and 13: various representations of local cohomology modules obtained in the earlier chapters inherit natural gradings when the ring, module and ideal concerned are all graded, and it seems to us that it is important to know that there is really only one sensible way of grading local cohomology modules in such circumstances. Our main aim in Chapter 12 has been to address this point. In Chapter 13, 'graded frills' are added to basic results proved earlier in the book.

The short Chapter 14 establishes some links between graded local cohomology and projective varieties; it has been included to provide a little geometric insight, and in order to motivate the work on Castelnuovo–Mumford regularity in Chapters 15–17, and the connections between ideal transforms and section functors of sheaves presented in Chapter 20.

In Chapter 15, we study the graded local cohomology of a homogeneous positively graded commutative Noetherian ring R with respect to the irrelevant ideal. One of the most important invariants in this context is Castelnuovo–Mumford regularity. This concept has, in addition to fundamental significance in projective algebraic geometry, connections with the degrees of generators of a finitely generated graded R-module M: it turns out that M can be generated by homogeneous elements of degrees not exceeding $\operatorname{reg}(M)$, the Castelnuovo–Mumford regularity of M. In turn, this leads on to connections with the theory of syzygies of finitely generated graded modules over polynomial rings over a field.

In certain circumstances, including when M is the vanishing ideal $I_{\mathbb{P}^r}(V)$ of a projective variety $V \subset \mathbb{P}^r$, the above-mentioned $\operatorname{reg}(M)$ coincides with $\operatorname{reg}^2(M)$, the Castelnuovo–Mumford regularity of M at and above level 2. In Chapters 16 and 17, we present bounds for the invariant $\operatorname{reg}^2(M)$. Chapter 16 contains *a priori* bounds which apply whenever the underlying homogeneous positively graded commutative Noetherian ring R has Artinian 0-th component. Chapter 17 is more specialized, and contains bounds expressed in terms of coefficients of Hilbert polynomials; our development of this theory includes a presentation of basic ideas concerning cohomological Hilbert polynomials. The motivation for our work in Chapter 17 comes from D. Mumford's classical work [54]: Mumford established the existence of bounds of the type we present, but, in the spirit of this book, we have added some precision.

One could view Chapters 18 and 19 as propaganda for the effectiveness of local cohomology as a tool in algebra and geometry. Chapter 18 presents some applications of Castelnuovo–Mumford regularities to reductions of ideals. This is a fast developing area, and we have not attempted to give an encyclopaedic account; instead, we have tried to present the basic ideas and a few recent results to whet the reader's appetite. The highlight of Chapter 18 is a theorem of L. T. Hoa; the statement of this theorem is satisfyingly simple, and makes no mention of local cohomology, and yet Hoa's proof, which we present towards the end of the chapter, makes significant use of graded local cohomology.

Chapter 18 is a good advertisement for local cohomology as a 'hidden tool', and Chapter 19 continues this theme, although here the applications (to the connectivity of algebraic varieties) are more geometrical in nature. The only appearances of local cohomology in Chapter 19 are in just two proofs, where a few central ideas (such as the Mayer–Vietoris Sequence and the Lichtenbaum–Hartshorne Vanishing Theorem) are used in crucial ways. No hypothesis or conclusion of any result in the chapter makes any mention of local cohomology, and yet we are able to show how the two results whose proofs use local cohomology can be developed into a theory which leads to proofs of major results involving connectivity, such as Grothendieck's Connectedness Theorem, the Bertini–Gothendieck Connectivity Theorem, the Connectedness Theorem for Projective Varieties due to W. Barth, to W. Fulton and J. Hansen, and to G. Faltings, and a ring-theoretic version of Zariski's Main Theorem. This chapter is certainly a good advertisement for the power of local cohomology as a tool in algebraic geometry!

Finally, in Chapter 20, we bring the subject 'home to its roots', so to speak, by presenting links between local cohomology and the cohomology of quasi-coherent sheaves over certain Noetherian schemes. (Chapter 20 is the only one for which we have assumed that the reader has some basic knowledge about schemes and sheaves.)

Some parts of our presentation are fairly leisurely: this is deliberate, and has been done with graduate students in mind, because we found several preparatory topics where either we knew of no suitable text-book account, or we felt we had something to add to the existing accounts. Examples are the treatments of Matlis duality in Chapter 10, of *canonical modules in Chapter 13, of reductions of ideals in Chapter 18, and of connectedness dimensions in Chapter 19; also, our presentation in Chapter 5 of some links between Koszul complexes and local cohomology is deliberately slow.

Our philosophy throughout has been to try to give a careful and accessible presentation of basic ideas and some important results, illustrating the ideas with examples, to bring the reader to a level of expertise where he or she can approach with some confidence recent research papers in local cohomology. To help with this, the book contains a large number of exercises, and we have supplied hints for many of the more difficult ones.

We have tried out parts of the book, especially the earlier chapters, on some of our own graduate students, and their comments have influenced the final version. We are particularly grateful to Claudia Albertini, Carlo Matteotti, Francesco Mordasini, Henrike Petzl and Massoud Tousi for acting as 'guinea pigs', so to speak. We should also like to express our gratitude to Peter Gabriel, John Greenlees, Martin Holland and Josef Rung for continual interest and encouragement, and to the Schweizerischer Nationalfonds zur Förderung der wissenschaftlichen Forschung, the Forschungsrat des Instituts für Mathematik der Universität Zürich, and the University of Sheffield Research Fund, for financial support to enable several visits for intense collaboration on the book to take place. Both authors would like to thank Alice Sharp: Markus Brodmann thanks her for kind hospitality during pleasant visits to Sheffield for discussions on the book; and Rodney Sharp thanks her for much sympathetic support through the years during which this book was being written (as well as for many things which have nothing to do with local cohomology). Finally, we are very grateful to David Tranah and Roger Astley of Cambridge University Press for their continual encouragement and assistance over many years, and, not least, for their cooperation over our request that the blue stripe on the cover of the book should match the blue of the Zürich trams!

Markus Brodmann — Rodney Sharp
Zürich — Sheffield

April 1997

Preface to the Second Edition

In the fifteen years since we completed the First Edition of this book, we have had opportunity to reflect on how we could change it in order to enhance its usefulness to the graduate students at whom it is aimed. As a result, this Second Edition shows substantial differences from the First. The main ones are described as follows.

One of the more dramatic changes is the introduction of a complete new chapter, Chapter 12, devoted to the study of canonical modules. The treatment of canonical modules in the First Edition was brief and restricted to the case where the underlying ring is Cohen–Macaulay; we assumed that the reader was familiar with the treatment in this case by W. Bruns and J. Herzog in their book on Cohen–Macaulay rings (see [7]). In our new Chapter 12, we present some of the basic work of Y. Aoyama (see [1] and [2]) and follow M. Hochster and C. Huneke [39] in defining a canonical module over a (not necessarily Cohen–Macaulay) local ring $(R, \mathfrak{m})$ to be a finitely generated R-module whose Matlis dual is isomorphic to the 'top' local cohomology module $H^{\dim R}_{\mathfrak{m}}(R)$. Thus this topic is intimately related to local cohomology.

Canonical modules have connections with the theory of S_2-ifications (here, the 'S_2' refers to Serre's condition), and we realised that the development of S_2-ifications can be facilitated by generalizations of arguments we had used to study ideal transforms in §2.2 of the First Edition. For this reason, instead of dealing just with ideal transforms based on the sequence of powers of a fixed ideal, we treat, in §2.2 of this Second Edition, a generalization based on a set $\mathfrak{B}$ of ideals such that, whenever $\mathfrak{b}, \mathfrak{c} \in \mathfrak{B}$, there exists $\mathfrak{d} \in \mathfrak{B}$ with $\mathfrak{d} \subseteq \mathfrak{bc}$. This represents a significant change to Chapter 2.

Another major change concerns our treatment of graded local cohomology. Our new Chapters 13 and 14 treat local cohomology in the situation where the rings, ideals and modules concerned are graded by $\mathbb{Z}^n$, where n is a positive integer. In the First Edition we dealt only with the case where $n = 1$; in the years since that edition was published, there have been more and more uses of local cohomology in multi-graded situations. It was not difficult for us to adapt the treatment of $\mathbb{Z}$-graded local cohomology from the First Edition to the multi-graded case. The main point of our Chapter 13 is to show that, even though there appear to be various possible approaches, there is really only one sensible way of grading local cohomology. Chapter 14 adds '($\mathbb{Z}^n$-)graded frills' to basic results proved earlier in the book. We illustrate this work with some calculations over polynomial rings and Stanley–Reisner rings. Chapters 13 and 14 are generalizations of Chapters 12 and 13 from the First Edition; however, we found it desirable to present some fundamental results from S. Goto's and K.-i. Watanabe's paper [22] about $\mathbb{Z}^n$-graded rings and modules; those results in the particular case when $n = 1$ are more readily available.

The last two decades have seen a surge in the use of local cohomology as a tool in 'characteristic p' commutative algebra, that is, the study of commutative Noetherian rings of prime characteristic p. The key to this is the fact that, for an ideal $\mathfrak{a}$ of such a ring R, and any integer $i \geq 0$, the i-th local cohomology module $H^i_{\mathfrak{a}}(R)$ of R itself with respect to $\mathfrak{a}$ has a so-called 'Frobenius action'. In the new §5.3, we explain why this Frobenius action exists, and in the new §6.5, we use it to present Hochster's proof of his Monomial Conjecture (in characteristic p), and to give some examples of how local cohomology can be used as an effective tool in tight closure theory.

Another new section is §20.5 about locally free sheaves; here we prove Serre's Cohomological Criterion for Local Freeness, Horrocks' Splitting Criterion and Grothendieck's Splitting Theorem. We have also expanded §20.4 with an additional application to projective schemes: we now include a result of Serre about the global generation of twisted coherent sheaves.

In order to include all this new material, we have had to omit some items that were included in the First Edition but which, we now consider, no longer command sufficiently compelling reasons for inclusion. The main topics that fall under this heading are the old §11.3 containing some applications of local duality (one can take the view that the new Chapter 12, on canonical modules, represents a major application of local duality), and the *a priori* bounds of diagonal type on Castelnuovo–Mumford regularity at and above level 2 that were treated in Chapter 16 of the First Edition (Chapter 17 has been reorganized to smooth over the omission, and expanded by the addition of further bounding results which follow from our generalized version of Mumford's bound on regularity at and above level 2).

There are also many minor changes, designed to improve the presentation or the usefulness of the book. For example, the treatment of Faltings' Annihilator Theorem in Chapter 9 now applies to two arbitrary ideals $\mathfrak{a}$ and $\mathfrak{b}$, whereas in the First Edition we treated only the case where $\mathfrak{b} \subseteq \mathfrak{a}$; the syzygetic characterization of Castelnuovo–Mumford regularity is given a full proof in the Second Edition; and the graded Deligne Isomorphism in §20.2 is presented here for a multi-graded situation.

We should also point out that two comments made about the First Edition in its Preface do not apply to this Second Edition. Firstly, the example studied in 2.3.7, 3.3.5, 4.3.7, ... is not called 'Hartshorne's Example' in this Second Edition (but we do cite Hartshorne's paper [28] when we first consider this example); secondly, there are a few more appearances of local cohomology in Chapter 19 than there were in the First Edition (because we followed a suggestion of M. Varbaro that, in some formulas, the arithmetic rank of an ideal $\mathfrak{a}$ could be replaced by the cohomological dimension of $\mathfrak{a}$). Nevertheless, Chapter 19 still contains several exciting examples of situations which represent 'hidden applications' of local cohomology, in the sense that significant results that do not mention local cohomology in either their hypotheses or their conclusions have proofs in this book that depend on local cohomology. There are other examples of such 'hidden applications' in §6.5 and in Chapter 18.

We would like to add, to the list of people thanked in the Preface to the First Edition, several more of our students, namely Roberto Boldini, Stefan Fumasoli, Simon Kurmann, Nicole Nossem and Fred Rohrer, who all contributed to this Second Edition, either by providing constructive criticism of the First Edition, or by trying out drafts of changed or new sections that we planned to include in the Second Edition. We are grateful to them all.

We thank the Schweizerischer Nationalfonds zur Förderung der wissenschaftlichen Forschung, the Forschungsrat des Instituts für Mathematik der Universität Zürich, and the Department of Pure Mathematics of the University of Sheffield, for financial support for visits for collaboration on this Second Edition. We are also particularly grateful to the Scientific Council of the Centre International de Rencontres Mathématiques (CIRM) at Luminy, Marseille, for their award to us of a two-week 'research in pairs' in Spring 2011 that enabled us, in the excellent environment for mathematical research at CIRM, to produce a complete draft. We again thank Alice Sharp for her continued support and encouragement for our project. It is also a pleasure for us to record our gratitude to Roger Astley and Clare Dennison of Cambridge University Press for their encouragement and support.

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April 2012

Notation and conventions

All rings considered in this book will have identity elements.

Throughout the book, R will always denote a non-trivial commutative Noetherian ring, and $\mathfrak{a}$ will denote an ideal of R. We shall only assume that R has additional properties (such as being local) when these are explicitly stated; however, the phrase '$(R, \mathfrak{m})$ is a local ring' will mean that R is a commutative Noetherian quasi-local ring with unique maximal ideal $\mathfrak{m}$.

For an ideal $\mathfrak{c}$ of R, we denote $\operatorname{Supp}(R/\mathfrak{c}) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{c}\}$ by $\operatorname{Var}(\mathfrak{c})$, and refer to this as *the variety of* $\mathfrak{c}$.

By a *multiplicatively closed subset* of R, we shall mean a subset of R which is closed under multiplication and contains 1. It should be noted (and this comment is relevant for the final chapter) that, if S is a non-empty subset of R which is closed under multiplication, then, even if S does not contain 1, we can form the commutative ring $S^{-1}R$ and, for an R-module M, the $S^{-1}R$-module $S^{-1}M$. In fact, $S^{-1}R \cong (S \cup \{1\})^{-1}R$, and, in $S^{-1}R$, the element sr/s, for $r \in R$ and $s \in S$, is independent of the choice of such s; similar comments apply to $S^{-1}M$.

The symbol $\mathbb{Z}$ will always denote the ring of integers; in addition, $\mathbb{N}$ (respectively $\mathbb{N}_0$) will always denote the set of positive (respectively non-negative) integers. The field of rational (respectively real, complex) numbers will be denoted by $\mathbb{Q}$ (respectively $\mathbb{R}$, $\mathbb{C}$).

The category of all modules and homomorphisms over a commutative ring R' will be denoted by $\mathcal{C}(R')$. When R' is G-graded, where G is a finitely generated, torsion-free Abelian group, the category of all graded R'-modules and homogeneous R'-homomorphisms will be denoted by $^*\mathcal{C}(R')$ (or $^*\mathcal{C}^G(R')$ when it is desirable to indicate the grading group G).

The symbol $\subseteq$ will stand for 'is a subset of'; the symbol $\subset$ will be reserved to denote strict inclusion. Thus, for sets A, B, the expression $A \subset B$ means that $A \subseteq B$ and $A \neq B$.

The identity mapping on a set A will be denoted by Id_A. If $f : A \to C$ is a mapping from the set A to the set C, and $S \subseteq A$, then $f\lceil_S : S \to C$ will denote the restriction of f to S. Thus $f\lceil_S(s) = f(s)$ for all $s \in S$.

Some of the exercises in the book are needed for the main development later in the book, and these exercises are marked with a '$\sharp$'; however, exercises which are used later in the book but only in other exercises have not been marked with a '$\sharp$'.