1

The local cohomology functors

The main objective of this chapter is to introduce the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ (throughout the book, $\mathfrak{a}$ always denotes an ideal in a (non-trivial) commutative Noetherian ring R) and its right derived functors $H^i_{\mathfrak{a}}$ $(i \geq 0)$, referred to as the local cohomology functors with respect to $\mathfrak{a}$. We shall see that $\Gamma_{\mathfrak{a}}$ is naturally equivalent to the functor $\varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^n, \bullet\,)$ and, indeed, that $H^i_{\mathfrak{a}}$ is naturally equivalent to the functor $\varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, \bullet\,)$ for each $i \geq 0$; moreover, as $\Gamma_{\mathfrak{a}}$ turns out to be left exact, the functors $\Gamma_{\mathfrak{a}}$ and $H^0_{\mathfrak{a}}$ are naturally equivalent.

This chapter also serves notice that our approach is based on fundamental techniques of homological commutative algebra, such as ones based on connected sequences of functors (see [71, pp. 212–214]): readers familiar with such ideas, and with the local cohomology functors, might like to just glance through this chapter and to move rapidly on to Chapter 2.

1.1 Torsion functors

1.1.1 Definition. For each R-module M, set $\Gamma_{\mathfrak{a}}(M) = \bigcup_{n\in\mathbb{N}}(0 :_M \mathfrak{a}^n)$, the set of elements of M which are annihilated by some power of $\mathfrak{a}$. Note that $\Gamma_{\mathfrak{a}}(M)$ is a submodule of M. For a homomorphism $f : M \longrightarrow N$ of R-modules, we have $f(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$, and so there is a mapping $\Gamma_{\mathfrak{a}}(f) : \Gamma_{\mathfrak{a}}(M) \longrightarrow \Gamma_{\mathfrak{a}}(N)$ which agrees with f on each element of $\Gamma_{\mathfrak{a}}(M)$.

It is clear that, if $g : M \to N$ and $h : N \to L$ are further homomorphisms of R-modules and $r \in R$, then $\Gamma_{\mathfrak{a}}(h \circ f) = \Gamma_{\mathfrak{a}}(h) \circ \Gamma_{\mathfrak{a}}(f)$, $\Gamma_{\mathfrak{a}}(f + g) = \Gamma_{\mathfrak{a}}(f) + \Gamma_{\mathfrak{a}}(g)$, $\Gamma_{\mathfrak{a}}(rf) = r\Gamma_{\mathfrak{a}}(f)$ and $\Gamma_{\mathfrak{a}}(\mathrm{Id}_M) = \mathrm{Id}_{\Gamma_{\mathfrak{a}}(M)}$. Thus, with these assignments, $\Gamma_{\mathfrak{a}}$ becomes a covariant, R-linear functor from $\mathcal{C}(R)$ to itself. (We say that a functor $T : \mathcal{C}(R) \longrightarrow \mathcal{C}(R)$ is *R-linear* precisely when it is additive and $T(rf) = rT(f)$ for all $r \in R$ and all homomorphisms f of R-modules.) We call $\Gamma_{\mathfrak{a}}$ the $\mathfrak{a}$-*torsion functor*.

1.1.2 ♯Exercise. Let $\mathfrak{b}$ be a second ideal of R. Show that

$$\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(M)) = \Gamma_{\mathfrak{a}+\mathfrak{b}}(M)$$

for each R-module M.

1.1.3 ♯Exercise. Let $\mathfrak{b}$ be a second ideal of R. Show that $\Gamma_{\mathfrak{a}} = \Gamma_{\mathfrak{b}}$ if and only if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.

(The notation ♯, attached to some exercises, is explained in the section of 'Notation and conventions' following the Preface to the Second Edition.)

1.1.4 Exercise. Suppose that the ideal $\mathfrak{b}$ of R is a reduction of $\mathfrak{a}$; that is, $\mathfrak{b} \subseteq \mathfrak{a}$ and there exists $s \in \mathbb{N}$ such that $\mathfrak{b}\mathfrak{a}^s = \mathfrak{a}^{s+1}$. Show that $\Gamma_{\mathfrak{a}} = \Gamma_{\mathfrak{b}}$.

1.1.5 Exercise. For a prime number p, find $\Gamma_{p\mathbb{Z}}(\mathbb{Q}/\mathbb{Z})$.

1.1.6 Lemma. *The* $\mathfrak{a}$-*torsion functor* $\Gamma_{\mathfrak{a}} : \mathcal{C}(R) \longrightarrow \mathcal{C}(R)$ *is left exact.*

Proof. Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be an exact sequence of R-modules and R-homomorphisms. We must show that

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(L) \xrightarrow{\Gamma_{\mathfrak{a}}(f)} \Gamma_{\mathfrak{a}}(M) \xrightarrow{\Gamma_{\mathfrak{a}}(g)} \Gamma_{\mathfrak{a}}(N)$$

is still exact. It is clear that $\Gamma_{\mathfrak{a}}(f)$ is a monomorphism and it follows immediately from 1.1.1 that $\Gamma_{\mathfrak{a}}(g) \circ \Gamma_{\mathfrak{a}}(f) = 0$, so that

$$\operatorname{Im}(\Gamma_{\mathfrak{a}}(f)) \subseteq \operatorname{Ker}(\Gamma_{\mathfrak{a}}(g)).$$

To prove the reverse inclusion, let $m \in \operatorname{Ker}(\Gamma_{\mathfrak{a}}(g))$. Thus $m \in \Gamma_{\mathfrak{a}}(M)$, so that there exists $n \in \mathbb{N}$ such that $\mathfrak{a}^n m = 0$, and $g(m) = 0$. Now there exists $l \in L$ such that $f(l) = m$, and our proof will be complete if we show that $l \in \Gamma_{\mathfrak{a}}(L)$. To achieve this, note that, for each $r \in \mathfrak{a}^n$, we have $f(rl) = rf(l) = rm = 0$, so that $rl = 0$ because f is a monomorphism. Hence $\mathfrak{a}^n l = 0$. □

The result of Lemma 1.1.6 will become transparent to many readers once we have covered a little more theory, and related the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ to a functor defined in terms of direct limits of 'Hom' modules. However, before we proceed in that direction, we are going to introduce, at this early stage, the fundamental definition of the local cohomology modules of an R-module M with respect to $\mathfrak{a}$.

1.2 Local cohomology modules

1.2.1 Definitions. For $i \in \mathbb{N}_0$, the i-th right derived functor of $\Gamma_{\mathfrak{a}}$ is denoted by $H^i_{\mathfrak{a}}$ and will be referred to as the *i-th local cohomology functor with respect to $\mathfrak{a}$*.

For an R-module M, we shall refer to $H^i_{\mathfrak{a}}(M)$, that is, the result of applying the functor $H^i_{\mathfrak{a}}$ to M, as the *i-th local cohomology module of M with respect to $\mathfrak{a}$*, and to $\Gamma_{\mathfrak{a}}(M)$ as the *$\mathfrak{a}$-torsion submodule of M*. We shall say that M is *$\mathfrak{a}$-torsion-free* precisely when $\Gamma_{\mathfrak{a}}(M) = 0$, and that M is *$\mathfrak{a}$-torsion* precisely when $\Gamma_{\mathfrak{a}}(M) = M$, that is, if and only if each element of M is annihilated by some power of $\mathfrak{a}$.

It is probably appropriate for us to stress the implications of the above definition at this point, and list some basic properties of the local cohomology modules.

1.2.2 Properties of local cohomology modules. Let M be an arbitrary R-module.

(i) To calculate $H^i_{\mathfrak{a}}(M)$, one proceeds as follows. Take an injective resolution

$$I^\bullet : 0 \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots \longrightarrow I^i \xrightarrow{d^i} I^{i+1} \longrightarrow \cdots$$

of M, so that there is an R-homomorphism $\alpha : M \longrightarrow I^0$ such that the sequence

$$0 \longrightarrow M \xrightarrow{\alpha} I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots \longrightarrow I^i \xrightarrow{d^i} I^{i+1} \longrightarrow \cdots$$

is exact. Apply the functor $\Gamma_{\mathfrak{a}}$ to the complex $I^\bullet$ to obtain

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(I^0) \xrightarrow{\Gamma_{\mathfrak{a}}(d^0)} \cdots \longrightarrow \Gamma_{\mathfrak{a}}(I^i) \xrightarrow{\Gamma_{\mathfrak{a}}(d^i)} \Gamma_{\mathfrak{a}}(I^{i+1}) \longrightarrow \cdots$$

and take the i-th cohomology module of this complex; the result,

$$\operatorname{Ker}(\Gamma_{\mathfrak{a}}(d^i))/\operatorname{Im}(\Gamma_{\mathfrak{a}}(d^{i-1})),$$

which, by a standard fact of homological algebra, is independent (up to R-isomorphism) of the choice of injective resolution $I^\bullet$ of M, is $H^i_{\mathfrak{a}}(M)$.

(ii) Since $\Gamma_{\mathfrak{a}}$ is covariant and R-linear, it is automatic that each local cohomology functor $H^i_{\mathfrak{a}}$ $(i \in \mathbb{N}_0)$ is again covariant and R-linear.

(iii) Since $\Gamma_{\mathfrak{a}}$ is left exact, $H^0_{\mathfrak{a}}$ is naturally equivalent to $\Gamma_{\mathfrak{a}}$. Thus, loosely, we can use this natural equivalence to identify these two functors.

(iv) The reader should be aware of the long exact sequence of local cohomology modules which results from a short exact sequence of R-modules and R-homomorphisms, and so we spell out the details here.

Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be an exact sequence of R-modules and R-homomorphisms. Then, for each $i \in \mathbb{N}_0$, there is a connecting homomorphism $H^i_{\mathfrak{a}}(N) \to H^{i+1}_{\mathfrak{a}}(L)$, and these connecting homomorphisms make the resulting long sequence

$$\begin{array}{lllll}
0 \longrightarrow & H^0_{\mathfrak{a}}(L) & \xrightarrow{H^0_{\mathfrak{a}}(f)} & H^0_{\mathfrak{a}}(M) & \xrightarrow{H^0_{\mathfrak{a}}(g)} & H^0_{\mathfrak{a}}(N) \\
\longrightarrow & H^1_{\mathfrak{a}}(L) & \xrightarrow{H^1_{\mathfrak{a}}(f)} & H^1_{\mathfrak{a}}(M) & \xrightarrow{H^1_{\mathfrak{a}}(g)} & H^1_{\mathfrak{a}}(N) \\
\longrightarrow & \cdots & & & & \cdots \\
\longrightarrow & H^i_{\mathfrak{a}}(L) & \xrightarrow{H^i_{\mathfrak{a}}(f)} & H^i_{\mathfrak{a}}(M) & \xrightarrow{H^i_{\mathfrak{a}}(g)} & H^i_{\mathfrak{a}}(N) \\
\longrightarrow & H^{i+1}_{\mathfrak{a}}(L) & \longrightarrow & \cdots & &
\end{array}$$

exact. The reader should also be aware of the 'natural' or 'functorial' properties of these long exact sequences: if

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \\
0 & \longrightarrow & L' & \xrightarrow{f'} & M' & \xrightarrow{g'} & N' & \longrightarrow & 0
\end{array}$$

is a commutative diagram of R-modules and R-homomorphisms with exact rows, then, for each $i \in \mathbb{N}_0$, we not only have a commutative diagram

$$\begin{array}{ccccc}
H^i_{\mathfrak{a}}(L) & \xrightarrow{H^i_{\mathfrak{a}}(f)} & H^i_{\mathfrak{a}}(M) & \xrightarrow{H^i_{\mathfrak{a}}(g)} & H^i_{\mathfrak{a}}(N) \\
\downarrow{\scriptstyle H^i_{\mathfrak{a}}(\lambda)} & & \downarrow{\scriptstyle H^i_{\mathfrak{a}}(\mu)} & & \downarrow{\scriptstyle H^i_{\mathfrak{a}}(\nu)} \\
H^i_{\mathfrak{a}}(L') & \xrightarrow{H^i_{\mathfrak{a}}(f')} & H^i_{\mathfrak{a}}(M') & \xrightarrow{H^i_{\mathfrak{a}}(g')} & H^i_{\mathfrak{a}}(N')
\end{array}$$

(simply because $H^i_{\mathfrak{a}}$ is a functor!), but we also have a commutative diagram

$$\begin{array}{ccc}
H^i_{\mathfrak{a}}(N) & \longrightarrow & H^{i+1}_{\mathfrak{a}}(L) \\
\downarrow{\scriptstyle H^i_{\mathfrak{a}}(\nu)} & & \downarrow{\scriptstyle H^{i+1}_{\mathfrak{a}}(\lambda)} \\
H^i_{\mathfrak{a}}(N') & \longrightarrow & H^{i+1}_{\mathfrak{a}}(L')
\end{array}$$

in which the horizontal maps are the appropriate connecting homomorphisms.

The following remark will be used frequently in applications. It is an easy consequence of Exercise 1.1.3 and the definition of local cohomology functors in 1.2.1.

1.2.3 Remark. Let $\mathfrak{b}$ be a second ideal of R such that $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$. Then $H^i_{\mathfrak{a}} = H^i_{\mathfrak{b}}$ for all $i \in \mathbb{N}_0$, so that $H^i_{\mathfrak{a}}(M) = H^i_{\mathfrak{b}}(M)$ for each R-module M and all $i \in \mathbb{N}_0$.

The next four exercises might help the reader to consolidate the properties of local cohomology modules listed in 1.2.2. The first three of these exercises (for which non-trivial results from commutative algebra about injective dimension over the relevant rings are very helpful) give a tiny foretaste of results about the vanishing of local cohomology modules which are central to the subject, and which will feature prominently later in the book.

1.2.4 Exercise. Show that, for every Abelian group (that is, $\mathbb{Z}$-module) G and for every $a \in \mathbb{Z}$, we have $H^i_{\mathbb{Z}a}(G) = 0$ for all $i \geq 2$.

1.2.5 Exercise. Suppose that $(R, \mathfrak{m})$ is a regular local ring of dimension d. Show that, for each R-module M, we have $H^i_{\mathfrak{a}}(M) = 0$ for all $i > d$.

1.2.6 Exercise. Suppose that $(R, \mathfrak{m})$ is a Gorenstein local ring (see, for example, Matsumura [50, p. 142]) of dimension d. Show that, for each finitely generated R-module M of finite projective dimension, we have $H^i_{\mathfrak{a}}(M) = 0$ for all $i > d$. (Here is a hint: use the fact [50, Theorem 18.1] that the injective dimension of R as an R-module is d, and then use induction on the projective dimension of M.)

The next exercise investigates the behaviour of local cohomology modules under fraction formation: its results show that, speaking loosely, the local cohomology functors ‘commute’ with fraction formation. This is a fundamental fact in the subject; however, we shall actually derive it as an immediate consequence of a more general result in Chapter 4 concerning the behaviour of local cohomology under flat base change (and we shall not make use of it until after Chapter 4). Nevertheless, even at this early stage, its proof should not present much difficulty for a reader familiar with the fact (proved in 10.1.14) that, if I is an injective R-module and S is a multiplicatively closed subset of R, then $S^{-1}I$ is an injective $S^{-1}R$-module.

1.2.7 Exercise. Let M be an R-module and let S be a multiplicatively closed subset of R. Show that $S^{-1}(\Gamma_{\mathfrak{a}}(M)) = \Gamma_{\mathfrak{a}S^{-1}R}(S^{-1}M)$, and that, for all $i \in \mathbb{N}_0$, there is an isomorphism of $S^{-1}R$-modules

$$S^{-1}(H^i_{\mathfrak{a}}(M)) \cong H^i_{\mathfrak{a}S^{-1}R}(S^{-1}M).$$

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More Information

It is now time for us to relate the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ to a functor defined in terms of direct limits of 'Hom' modules. Fundamental to the discussion is the natural isomorphism, for an R-module M and $n \in \mathbb{N}$,

$$\phi := \phi_{\mathfrak{a}^n,M} : \operatorname{Hom}_R(R/\mathfrak{a}^n, M) \xrightarrow{\cong} (0 :_M \mathfrak{a}^n)$$

for which $\phi(f) = f(1 + \mathfrak{a}^n)$ for all $f \in \operatorname{Hom}_R(R/\mathfrak{a}^n, M)$. In fact, we are going to put the various $\phi_{\mathfrak{a}^n,M}$ $(n \in \mathbb{N})$ together to obtain a natural isomorphism $\varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^n, M) \xrightarrow{\cong} \Gamma_{\mathfrak{a}}(M)$, but before we do this it might be helpful to the reader if we give some general considerations about functors and direct limits, as the principles involved will be used numerous times in this book.

1.2.8 Remarks. Let $(\Lambda, \leq)$ be a (non-empty) directed partially ordered set, and suppose that we are given an inverse system of R-modules $(W_\alpha)_{\alpha\in\Lambda}$ over Λ, with constituent R-homomorphisms $h^\alpha_\beta : W_\alpha \to W_\beta$ (for each $(\alpha, \beta) \in \Lambda \times \Lambda$ with $\alpha \geq \beta$). Let $T : \mathcal{C}(R) \times \mathcal{C}(R) \to \mathcal{C}(R)$ be an R-linear functor of two variables which is contravariant in the first variable and covariant in the second. (A functor $U : \mathcal{C}(R) \times \mathcal{C}(R) \longrightarrow \mathcal{C}(R)$ is said to be R-*linear* precisely when it is additive and $U(rf, g) = rU(f, g) = U(f, rg)$ for all $r \in R$ and all homomorphisms f, g of R-modules.) We show now how these data give rise to a covariant, R-linear functor

$$\varinjlim_{\alpha\in\Lambda} T(W_\alpha, \bullet) : \mathcal{C}(R) \longrightarrow \mathcal{C}(R).$$

Let M, N be R-modules and let $f : M \longrightarrow N$ be an R-homomorphism. For $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$, the homomorphism $h^\alpha_\beta : W_\alpha \longrightarrow W_\beta$ induces an R-homomorphism

$$T(h^\alpha_\beta, M) : T(W_\beta, M) \longrightarrow T(W_\alpha, M),$$

and the fact that T is a functor ensures that the $T(h^\alpha_\beta, M)$ turn the family $(T(W_\alpha, M))_{\alpha\in\Lambda}$ into a direct system of R-modules and R-homomorphisms over Λ. We may therefore form $\varinjlim_{\alpha\in\Lambda} T(W_\alpha, M)$. Moreover, again for $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$, we have a commutative diagram

$$\begin{array}{ccc} T(W_\beta, M) & \xrightarrow{T(h^\alpha_\beta, M)} & T(W_\alpha, M) \\ \downarrow{\scriptstyle T(W_\beta, f)} & & \downarrow{\scriptstyle T(W_\alpha, f)} \\ T(W_\beta, N) & \xrightarrow{T(h^\alpha_\beta, N)} & T(W_\alpha, N) \end{array} \quad ;$$

therefore the $T(W_\alpha, f)$ $(\alpha \in \Lambda)$ constitute a morphism of direct systems and so induce an R-homomorphism

$$\varinjlim_{\alpha\in\Lambda} T(W_\alpha, f) : \varinjlim_{\alpha\in\Lambda} T(W_\alpha, M) \longrightarrow \varinjlim_{\alpha\in\Lambda} T(W_\alpha, N).$$

It is now straightforward to check that, in this way, $\varinjlim_{\alpha\in\Lambda} T(W_\alpha, \bullet\,)$ becomes a covariant, R-linear functor from $\mathcal{C}(R)$ to itself. Observe that, since passage to direct limits preserves exactness, if T is left exact, then so too is this new functor.

1.2.9 Examples. Here we present some examples that are central for our subject.

(i) Probably the most important examples for us of the ideas of 1.2.8 concern the case where we take for Λ the set $\mathbb{N}$ of positive integers with its usual ordering and the inverse system $(R/\mathfrak{a}^n)_{n\in\mathbb{N}}$ of R-modules under the natural homomorphisms $h^n_m : R/\mathfrak{a}^n \to R/\mathfrak{a}^m$ (for $n, m \in \mathbb{N}$ with $n \geq m$) (in such circumstances, $\mathfrak{a}^n \subseteq \mathfrak{a}^m$, of course). In this way, we obtain covariant, R-linear functors

$$\varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^n, \bullet\,) \quad \text{and} \quad \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, \bullet\,) \;\; (i \in \mathbb{N}_0)$$

from $\mathcal{C}(R)$ to itself. Of course, the natural equivalence between the left exact functors Hom_R and Ext^0_R leads to a natural equivalence between the left exact functors

$$\varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^n, \bullet\,) \quad \text{and} \quad \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^0_R(R/\mathfrak{a}^n, \bullet\,)$$

which we shall use without further comment.

(ii) Very similar considerations, this time based on the inclusion maps $\mathfrak{a}^n \to \mathfrak{a}^m$ (for $n, m \in \mathbb{N}$ with $n \geq m$), lead to functors (which are again covariant and R-linear)

$$\varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(\mathfrak{a}^n, \bullet\,) \quad \text{and} \quad \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(\mathfrak{a}^n, \bullet\,) \;\; (i \in \mathbb{N}_0)$$

from $\mathcal{C}(R)$ to itself, and a natural equivalence between the left exact functors

$$\varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(\mathfrak{a}^n, \bullet\,) \quad \text{and} \quad \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^0_R(\mathfrak{a}^n, \bullet\,).$$

These functors will be considered in detail in Chapter 2.

It will be convenient for us to consider situations slightly more general than that studied in 1.2.9(i) above.

1.2.10 Definition and Example. Let $(\Lambda, \leq)$ be a (non-empty) directed partially ordered set. By an *inverse family of ideals (of R) over* Λ, we mean a family $(\mathfrak{b}_\alpha)_{\alpha\in\Lambda}$ of ideals of R such that, whenever $(\alpha, \beta) \in \Lambda \times \Lambda$ with $\alpha \geq \beta$, we have $\mathfrak{b}_\alpha \subseteq \mathfrak{b}_\beta$.

For example, if

$$\mathfrak{b}_1 \supseteq \mathfrak{b}_2 \supseteq \cdots \supseteq \mathfrak{b}_n \supseteq \mathfrak{b}_{n+1} \supseteq \cdots$$

is a descending chain of ideals of R, then $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ is an inverse family of ideals over $\mathbb{N}$ (with its usual ordering). In particular, the family $(\mathfrak{a}^n)_{n\in\mathbb{N}}$ is an inverse family of ideals over $\mathbb{N}$.

Let $(\mathfrak{b}_\alpha)_{\alpha\in\Lambda}$ be an inverse family of ideals of R over Λ. Then the natural R-homomorphisms $h^\alpha_\beta : R/\mathfrak{b}_\alpha \to R/\mathfrak{b}_\beta$ (for $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$) turn $(R/\mathfrak{b}_\alpha)_{\alpha\in\Lambda}$ into an inverse system over Λ, and so we can apply the ideas of 1.2.8 to produce covariant, R-linear functors

$$\varinjlim_{\alpha\in\Lambda} \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, \bullet\,) \quad \text{and} \quad \varinjlim_{\alpha\in\Lambda} \operatorname{Ext}^i_R(R/\mathfrak{b}_\alpha, \bullet\,) \ \ (i \in \mathbb{N}_0)$$

(from $\mathcal{C}(R)$ to itself), the first two of which are left exact and naturally equivalent.

1.2.11 Theorem. *Let $\mathfrak{B} = (\mathfrak{b}_\alpha)_{\alpha\in\Lambda}$ be an inverse family of ideals of R over Λ, as in* 1.2.10.

(i) *There is a covariant, R-linear functor $\Gamma_\mathfrak{B} : \mathcal{C}(R) \to \mathcal{C}(R)$ which is such that, for an R-module M,*

$$\Gamma_\mathfrak{B}(M) = \bigcup_{\alpha\in\Lambda} (0 :_M \mathfrak{b}_\alpha),$$

and, for a homomorphism $f : M \longrightarrow N$ of R-modules, $\Gamma_\mathfrak{B}(f) : \Gamma_\mathfrak{B}(M) \longrightarrow \Gamma_\mathfrak{B}(N)$ is just the restriction of f to $\Gamma_\mathfrak{B}(M)$.

(ii) *There is a natural equivalence*

$$\phi' \ (= \phi'_\mathfrak{B}) : \varinjlim_{\alpha\in\Lambda} \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, \bullet\,) \xrightarrow{\cong} \Gamma_\mathfrak{B}$$

(of functors from $\mathcal{C}(R)$ to itself) which is such that, for an R-module M and $\alpha \in \Lambda$, the image under ϕ'_M of the natural image of an $h \in \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, M)$ is $h(1 + \mathfrak{b}_\alpha)$. Consequently, $\Gamma_\mathfrak{B}$ is left exact.

(iii) *In particular, there is a natural equivalence*

$$\phi^0\ (= \phi^0_{\mathfrak{a}}) : \varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^n, \ \bullet\ ) \xrightarrow{\cong} \Gamma_{\mathfrak{a}}$$

which is such that, for an R-module M and $n \in \mathbb{N}$, the image under ϕ^0_M of the natural image of an $h \in \operatorname{Hom}_R(R/\mathfrak{a}^n, M)$ is $h(1+\mathfrak{a}^n)$.

Proof. (i) This can be proved by straightforward modification of the ideas of 1.1.1, and so will be left to the reader.

(ii) Let $f : M \longrightarrow N$ be a homomorphism of R-modules. For each $\alpha \in \Lambda$, let $\phi_{\mathfrak{b}_\alpha,M} : \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, M) \longrightarrow (0 :_M \mathfrak{b}_\alpha)$ be the R-isomorphism for which $\phi_{\mathfrak{b}_\alpha,M}(h) = h(1+\mathfrak{b}_\alpha)$ for all $h \in \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, M)$. Let $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$, and let $h^\alpha_\beta : R/\mathfrak{b}_\alpha \to R/\mathfrak{b}_\beta$ be as in 1.2.10. Since the diagram

$$\begin{array}{ccc} \operatorname{Hom}_R(R/\mathfrak{b}_\beta, M) & \xrightarrow[\cong]{\phi_{\mathfrak{b}_\beta,M}} & (0 :_M \mathfrak{b}_\beta) \\ {\scriptstyle \operatorname{Hom}_R(h^\alpha_\beta, M)}\downarrow & & \downarrow \\ \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, M) & \xrightarrow[\cong]{\phi_{\mathfrak{b}_\alpha,M}} & (0 :_M \mathfrak{b}_\alpha) \end{array}$$

(in which the right-hand vertical map is inclusion) commutes, it follows that there is indeed an R-isomorphism

$$\phi'_M : \varinjlim_{\alpha\in\Lambda} \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, M) \xrightarrow{\cong} \Gamma_{\mathfrak{B}}(M) = \bigcup_{\alpha\in\Lambda} (0 :_M \mathfrak{b}_\alpha)$$

as described in the statement of the theorem. It is easy to check that the diagram

$$\begin{array}{ccc} \varinjlim_{\alpha\in\Lambda} \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, M) & \xrightarrow[\cong]{\phi'_M} & \Gamma_{\mathfrak{B}}(M) \\ {\scriptstyle \varinjlim_{\alpha\in\Lambda} \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, f)}\downarrow & & \downarrow{\scriptstyle \Gamma_{\mathfrak{B}}(f)} \\ \varinjlim_{\alpha\in\Lambda} \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, N) & \xrightarrow[\cong]{\phi'_N} & \Gamma_{\mathfrak{B}}(N) \end{array}$$

commutes, and the final claim is then immediate from 1.2.10.

(iii) This is immediate from (ii), since when we apply (ii) to the family of ideals $\mathfrak{B} := (\mathfrak{a}^n)_{n\in\mathbb{N}}$, the functor $\Gamma_{\mathfrak{B}}$ of (i) is just the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$. □

We commented earlier that it would in time become transparent that $\Gamma_{\mathfrak{a}}$ is left exact: we had 1.2.11 in mind when we made that comment.

1.2.12 ♯Exercise. Provide a proof for part (i) of 1.2.11.

1.3 Connected sequences of functors

In this section, we are going to use the concepts of 'connected sequence of functors' and 'strongly connected sequence of functors'. These are explained on p. 212 of Rotman's book [71]. For the reader's convenience, we recall here relevant definitions in the case of negative connected sequences, as we shall be particularly concerned with this case.

1.3.1 Definition. Let R' be a commutative ring.

A sequence $(T^i)_{i\in\mathbb{N}_0}$ of covariant functors from $\mathcal{C}(R)$ to $\mathcal{C}(R')$ is said to be a *negative connected sequence* (respectively, a *negative strongly connected sequence*) if the following conditions are satisfied.

(i) Whenever $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is an exact sequence in $\mathcal{C}(R)$, there are defined connecting R'-homomorphisms

$$T^i(N) \longrightarrow T^{i+1}(L) \quad \text{for all } i \in \mathbb{N}_0$$

such that the long sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & T^0(L) & \xrightarrow{T^0(f)} & T^0(M) & \xrightarrow{T^0(g)} & T^0(N) \\ & \longrightarrow & T^1(L) & \xrightarrow{T^1(f)} & T^1(M) & \xrightarrow{T^1(g)} & T^1(N) \\ & \longrightarrow & \cdots & & & & \cdots \\ & \longrightarrow & T^i(L) & \xrightarrow{T^i(f)} & T^i(M) & \xrightarrow{T^i(g)} & T^i(N) \\ & \longrightarrow & T^{i+1}(L) & \longrightarrow & \cdots & & \end{array}$$

is a complex (respectively, is exact).

(ii) Whenever

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} & & \\ 0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & 0 \end{array}$$

is a commutative diagram of R-modules and R-homomorphisms with exact rows, then there is induced, by λ, μ and ν, a chain map of the long complex of (i) for the top row into the corresponding long complex for the bottom row.

It might help if we remind the reader of the convention regarding the raising and lowering of indices in a situation such as that of 1.3.1, under which