1
Orthonormal systems and their applications

In this chapter we introduce the main notion of analysis over a local field K of positive characteristic, and study two most important orthonormal systems appearing in this theory, the Carlitz polynomials and hyperdifferentiations. Their role, especially that of the Carlitz polynomials, will be crucial throughout this book. However the first applications, to the Carlitz module expressing the main functional relation for the counterparts of the exponential and logarithm, and to representations over K of the canonical commutation relations of quantum mechanics, will be given in this chapter. We also describe the digit principle connecting the analysis of additive functions and that of general continuous functions on subsets of K.

1.1 Basic notions

1.1.1. Function fields. Factorials. It is well known [17, 38, 122] that any local field (= nondiscrete locally compact topological field) of positive characteristic p is isomorphic to the field K of formal Laurent series with coefficients from the Galois field $\mathbb{F}_q$ of q elements, $q = p^\upsilon$, $\upsilon \in \mathbb{N}$. There is an absolute value $|\cdot|$ on K defined as follows: $|0| = 0$; if $z \in K$,

$$z = \sum_{i=m}^{\infty} \zeta_i x^i, \quad m \in \mathbb{Z},\ \zeta_i \in \mathbb{F}_q,\ \zeta_m \neq 0,$$

then $|z| = q^{-m}$. The absolute value has the following properties:

$$|z_1 z_2| = |z_1| \cdot |z_2|, \quad |z_1 + z_2| \leq \max(|z_1|, |z_2|), \quad z_1, z_2 \in K.$$

The second property, called *the ultra-metric inequality* (or *the non-Archimedean property*), implies that K is totally disconnected in the topology defined by the metric $|x-y|$. Here we encounter all the peculiar properties of non-Archimedean spaces (see, for example, [98]) – standard subsets of K, like balls or spheres, are simultaneously open and closed (in fact, compact); two balls either do not intersect, or one of them is contained in the other, a series $\sum\limits_{k=1}^{\infty} a_k$, $a_k \in K$, converges if and only if $a_k \to 0$, etc.

We will denote by $\overline{K}$ a fixed algebraic closure of K. The absolute value $|\cdot|$ can be extended in a unique way onto $\overline{K}$, and the completion $\overline{K}_c$ is algebraically closed (see [98]).

The first "appropriate" analog of the factorial is the sequence

$$D_i = [i][i-1]^q \dots [1]^{q^{i-1}}, \quad [i] = x^{q^i} - x \ (i \geq 1),\ D_0 = 1, \tag{1.1}$$

Another factorial-like sequence is

$$L_i = [i][i-1]\dots[1] \ (i \geq 1),\ L_0 = 1. \tag{1.2}$$

It is easy to see that

$$|D_i| = q^{-\frac{q^i-1}{q-1}}, \quad |L_i| = q^{-i}. \tag{1.3}$$

As we will see, in some cases it is natural to consider D_i as a counterpart of the factorial of the number q^i. Then analogs of other values of the factorial are defined as follows. Let us write any natural number j in the base q as

$$j = \sum_{i=0}^{n-1} \alpha_i q^i, \quad \alpha_i \in \mathbb{Z},\ 0 \leq \alpha_i < q.$$

Then we set

$$\Gamma_j = \prod_{i=0}^{n-1} D_i^{\alpha_i}. \tag{1.4}$$

Lemma 1.1 (i) *The elements D_m, L_m, and Γ_{q^m-1} are connected by the identity*

$$\Gamma_{q^m-1} L_m = D_m.$$

(ii) *For any $i = 1, 2, \dots$, the element $[i]$ is the product of all monic irreducible polynomials $m \in \mathbb{F}_q[x]$, such that $\deg m$ divides i.*

(iii) *For any* $i = 1, 2, \dots,$ *the element* D_i *is the product of all monic polynomials* $g \in \mathbb{F}_q[x]$ *with* $\deg g = i$.

(iv) L_i *is the least common multiple of all polynomials from* $\mathbb{F}_q$ *of degree* i.

Proof. (i) By the definition,

$$\Gamma_{q^m-1} = (D_0 \dots D_{m-1})^{q-1}.$$

We have $D_m = \prod_{i=1}^{m} [i]^{q^{m-i}}$, $L_m = \prod_{i=1}^{m} [i]$,

$$D_0 \dots D_{m-1} = \prod_{j=1}^{m-1} \prod_{i=1}^{j} [i]^{q^{j-i}} = \prod_{i=1}^{m-1} \prod_{j=i}^{m-1} [i]^{q^{j-i}}$$
$$= \prod_{i=1}^{m-1} [i]^{\sum_{j=i}^{m-1} q^{j-i}} = \prod_{i=1}^{m-1} [i]^{\frac{q^{m-i}-1}{q-1}},$$

so that

$$\Gamma_{q^m-1} L_m = [m] \prod_{i=1}^{m-1} [i]^{q^{m-i}} = D_m.$$

(ii) Let $m \in \mathbb{F}_q[x]$ be a monic irreducible polynomial. Then m divides $[i] = x^{q^i} - x$ if and only if $\deg m$ divides i.

Indeed, suppose that m divides $[i]$. Let α be a root of m lying in the splitting field of m. Then $\alpha^{q^i} = \alpha$, so that $\alpha \in \mathbb{F}_{q^i}$. Therefore the simple extension $\mathbb{F}_q(\alpha)$ of the field $\mathbb{F}_q$ is a subfield of $\mathbb{F}_{q^i}$. Since m is irreducible, we have $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = \deg m$, and it follows from the basic properties of finite fields [75, 76] that $\deg m$ divides $i = [\mathbb{F}_{q^i} : \mathbb{F}_q]$.

Conversely, if the number $l = \deg m$ divides i, then [75, 76] $\mathbb{F}_{q^l}$ is a subfield of $\mathbb{F}_{q^i}$. If α is a root of m in its splitting field over $\mathbb{F}_q$, then $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = l$, whence $\mathbb{F}_q(\alpha) = \mathbb{F}_{q^l}$. Therefore $\alpha \in \mathbb{F}_{q^i}$, so that $\alpha^{q^i} = \alpha$, and α is a root of the polynomial $x^{q^i} - x$. Since m is irreducible, it generates the principal ideal in $\mathbb{F}_q[x]$ consisting of polynomials for which α is a root. This means that m divides $x^{q^i} - x$.

Now it remains to note that $\frac{d}{dx}[i] = -1$, so that $[i] = x^{q^i} - x$ has no multiple roots in its splitting field over $\mathbb{F}_q$. Therefore each monic irreducible polynomial appears exactly once in the canonical decomposition of $[i]$ in the ring $\mathbb{F}_q[x]$.

(iii) Let $\pi \in \mathbb{F}_q[x]$ be an irreducible monic polynomial, $\deg \pi = d$. Among all monic polynomials $g \in \mathbb{F}_q[x]$ of degree $i \geq d$, q^{i-d} polynomials are multiples of π, q^{i-2d} $(i \geq 2d)$ polynomials are multiples of π^2, etc. Thus the number of polynomials of degree i, whose canonical decompositions contain π with exactly the power 1, equals $q^{i-d} - q^{i-2d}$. We can perform similar evaluations with higher degrees of π, so that the product of all monic polynomials of the degree i contains π with the power equal to

$$\left(q^{i-d} - q^{i-2d}\right) + 2\left(q^{i-2d} - q^{i-3d}\right) + \cdots + \operatorname{int}\left(\frac{i}{d}\right) q^{i-\operatorname{int}\left(\frac{i}{d}\right)d} = \sum_{\nu=1}^{\operatorname{int}\left(\frac{i}{d}\right)} q^{i-\nu d}$$

where $\operatorname{int}(s)$, $s \in \mathbb{R}^+$, means the biggest integer not exceeding s.

On the other hand, considering D_i we see that π is contained once in the canonical decomposition of $[j]$ exactly for $j = \nu d$, $\nu \leq \operatorname{int}\left(\frac{i}{d}\right)$. Since $D_i = \prod\limits_{j=1}^{i} [j]^{q^{i-j}}$, we find that π is contained in the canonical decomposition of D_i with the power $\sum\limits_{\nu=1}^{\operatorname{int}\left(\frac{i}{d}\right)} q^{i-\nu d}$, the same as the above one.

(iv) Reasoning similar to (iii) shows that both elements to be proved equal contain π with the same power $\operatorname{int}\left(\frac{i}{d}\right)$. ■

1.1.2. $\mathbb{F}_q$-linear functions. A special feature of the field K is the availability of many functions f with the property of $\mathbb{F}_q$-*linearity*

$$f(t_1 + t_2) = f(t_1) + f(t_2), \quad f(\xi t) = \xi f(t),$$

where t, t_1, t_2 belong to a $\mathbb{F}_q$-subspace of K, $\xi \in \mathbb{F}_q$. Such are, for example, all power series $\sum c_k t^{q^k}$, $c_k \in \overline{K}_c$, convergent on some region in K or $\overline{K}_c$.

In particular, let us consider $\mathbb{F}_q$-linear polynomials. Such a polynomial over $\overline{K}_c$ (and over any infinite field) has the form $\sum c_k t^{q^k}$ [45]. Obviously, the roots of an $\mathbb{F}_q$-linear polynomial form an $\mathbb{F}_q$-vector space. For separable polynomials, the converse is also true.

Proposition 1.2 *A separable polynomial $P \in \overline{K}_c[t]$ is $\mathbb{F}_q$-linear if and only if the set $W = \{w_1, \dots, w_m\}$ of all its roots forms an $\mathbb{F}_q$-vector subspace of $\overline{K}_c$*

Proof. The necessity is obvious, and we prove the sufficiency. We have

$$P(t) = \prod_{i=1}^{m} (t - w_i).$$

It is clear that $P(t+w) = P(t)$ for any $w \in W$. Suppose that $y \in \overline{K}_c$. Set

$$H(t) = P(t+y) - P(t) - P(y).$$

Then $\deg H < \deg P = m$, and at the same time $H(w) = 0$ for all $w \in W$. This means that $H(t) = 0$ for all t, so that H is additive.

Similarly, for $\alpha \in \mathbb{F}_q$ set

$$H_\alpha(t) = P(\alpha t) - \alpha P(t).$$

Taking a basis in the finite $\mathbb{F}_q$-vector space W we see that $m = q^l$ for some $l \in \mathbb{Z}_+$. Then $\deg P = q^l$. Since $\alpha^{q^l} = \alpha$, we conclude that $\deg H_\alpha < q^l$. On the other hand, $H_\alpha(w) = 0$ for $w \in W$. Therefore $H_\alpha(t) \equiv 0$, as desired. ■

A detailed exposition of algebraic properties of $\mathbb{F}_q$-linear polynomials is given in [45]. Here we describe some properties of $\mathbb{F}_q$-linear power series convergent on a neighborhood of the origin.

Let $\mathcal{R}_K$ be the set of all formal power series $a = \sum\limits_{k=0}^{\infty} a_k t^{q^k}$ where $a_k \in K$, $|a_k| \le A^{q^k}$, and A is a positive constant depending on a. In fact each series $a = a(t)$ from $\mathcal{R}_K$ converges on a neighborhood of the origin in K (and $\overline{K}_c$).

$\mathcal{R}_K$ is a ring with respect to the termwise addition and the composition

$$a \circ b = \sum_{l=0}^{\infty} \left(\sum_{n=0}^{l} a_n b_{l-n}^{q^n} \right) t^{q^l}, \quad b = \sum_{k=0}^{\infty} b_k t^{q^k},$$

as the operation of multiplication. Indeed, if $|b_k| \le B^{q^k}$, then, by the ultra-metric property of the absolute value,

$$\left| \sum_{n=0}^{l} a_n b_{l-n}^{q^n} \right| \le \max_{0 \le n \le l} A^{q^n} \left(B^{q^{l-n}} \right)^{q^n} \le C^{q^l}$$

where $C = B \max(A, 1)$. The unit element in $\mathcal{R}_K$ is $a(t) = t$. It is easy to check that $\mathcal{R}_K$ has no zero divisors.

If $a \in \mathcal{R}_K$, $a = \sum\limits_{k=0}^{\infty} a_k t^{q^k}$, is such that $|a_0| \le 1$ and $|a_k| \le A^{q^k}$, $|A| \ge 1$, for all k, then we may write

$$|a_k| \le A_1^{q^k - 1}, \quad k = 0, 1, 2, \dots,$$

if we take $A_1 \ge A^{q^k/(q^k-1)}$ for all $k \ge 1$. If also $b = \sum\limits_{k=0}^{\infty} b_k t^{q^k}$, $|b_k| \le B_1^{q^k-1}$,

$B_1 \geq 1$, then for $a \circ b = \sum\limits_{l=0}^{\infty} c_l t^{q^l}$ we have

$$|c_l| \leq \max_{i+j=l} A_1^{q^i-1} \left(B_1^{q^j-1} \right)^{q^i} \leq C_1^{q^l-1}$$

where $C_1 = \max(A_1, B_1)$. In particular, in this case the coefficients of the series for a^n (the composition power) satisfy an estimate of this kind, with a constant independent of n.

Proposition 1.3 *The ring $\mathcal{R}_K$ is a left Ore ring, thus it possesses a classical ring of fractions.*

Proof. By Ore's theorem (see [50]) it suffices to show that for any elements $a, b \in \mathcal{R}_K$ there exist elements $a', b' \in \mathcal{R}_K$ such that $b' \neq 0$ and

$$a' \circ b = b' \circ a. \tag{1.5}$$

We may assume that $a \neq 0$,

$$a = \sum_{k=m}^{\infty} a_k t^{q^k}, \quad b = \sum_{k=l}^{\infty} b_k t^{q^k},$$

$m, l \geq 0$, $a_m \neq 0$, $b_l \neq 0$.

Without restricting generality we may assume that $l = m$ (if we prove (1.5) for this case and if, for example, $l < m$, we set $b_1 = t^{q^{m-l}} \circ b$, find a'', b' in such a way that $a'' \circ b_1 = b' \circ a$, and then set $a' = a'' \circ t^{q^{m-l}}$), and that $a_l = b_l = \alpha$, so that

$$a = \alpha t^{q^l} + \sum_{k=l+1}^{\infty} a_k t^{q^k}, \quad b = \alpha t^{q^l} + \sum_{k=l+1}^{\infty} b_k t^{q^k},$$

$\alpha \neq 0$.

We seek a', b' in the form

$$a' = \sum_{j=l}^{\infty} a'_j t^{q^j}, \quad b' = \sum_{j=l}^{\infty} b'_j t^{q^j}.$$

The coefficients a'_j, b'_j can be defined inductively. Set $a'_l = b'_l = 1$. If a'_j, b'_j have been determined for $l \leq j \leq k-1$, then a'_k, b'_k are determined from the equality of the $(k+l)$-th terms of the composition products:

$$a'_k \alpha^{q^k} + \sum_{\substack{i+j=k+l \\ j \neq l}} a'_i b_j^{q^i} = b'_k \alpha^{q^k} + \sum_{\substack{i+j=k+l \\ j \neq l}} b'_i a_j^{q^i}$$

(the above sums do not contain nontrivial terms with a'_i, b'_i, $i \geq k$, since $a_j = b_j = 0$ for $j < l$).

In particular, we may set $b'_k = 0$,

$$a'_k = \alpha^{-q^k} \left\{ \sum_{\substack{i+j=k+l \\ i<k,\, j\neq l}} \left(a'_i b_j^{q^i} - b'_i a_j^{q^i} \right) \right\}.$$

If this choice is made for each $k \geq l+1$, then we have $b'_i = 0$ for every $i \geq l+1$, so that

$$a'_k = \alpha^{-q^k} \sum_{\substack{i+j=k+l \\ i<k,\, j\neq l}} a'_i b_j^{q^i}. \tag{1.6}$$

Denote $C_1 = |\alpha|^{-1}$. We have $|b_j| \leq C_2^{q^j}$ for all j. Denote, further, $C_3 = \max(1, C_1, C_2)$, $C_4 = C_3^{q^{l+2}}$. Let us prove that

$$|a'_k| \leq C_4^{q^k}.$$

Suppose that $|a'_i| \leq C_4^{q^i}$ for all i, $l \leq i \leq k-1$ (this is obvious for $i = l$, since $a'_l = 1$). By (1.6),

$$\begin{aligned} |a'_k| &\leq C_1^{q^k} \max_{\substack{i+j=k+l \\ i<k,\, j\neq l}} C_4^{q^i} C_2^{q^{i+j}} \leq C_1^{q^k} C_4^{q^{k-1}} C_2^{q^{k+l}} \\ &\leq C_3^{q^k + q^{k+l+1} + q^{k+l}} = C_3^{(1+q^l+q^{l+1})q^k} \leq \left(C_3^{q^{l+2}} \right)^{q^k} = C_4^{q^k}, \end{aligned}$$

as desired. Thus $a' \in \mathcal{R}_K$. ■

Every nonzero element of $\mathcal{R}_K$ is invertible in the ring of fractions $\mathcal{A}_K$, which is actually a skew field consisting of formal fractions $c^{-1}d$, $c, d \in \mathcal{R}_K$.

Proposition 1.4 *Each element $a = c^{-1}d \in \mathcal{A}_K$ can be represented in the form $a = t^{q^{-m}} a'$ where $t^{q^{-m}}$ is the inverse of t^{q^m}, $a' \in \mathcal{R}_K$.*

Proof. It is sufficient to prove that any nonzero element $c \in \mathcal{R}_K$ can be written as $c = c' \circ t^{q^m}$ where c is invertible in $\mathcal{R}_K$.

Let $c = \sum_{k=m}^{\infty} c_k t^{q^k}$, $c_m \neq 0$, $|c_k| \leq C^{q^k}$. Then

$$c = c_m \left(t + \sum_{l=1}^{\infty} c_m^{-1} c_{m+l} t^{q^l} \right) \circ t^{q^m}$$

where $\left|c_m^{-1}c_{m+l}\right| \le C_1^{q^l-1}$ for all $l \ge 1$, if C_1 is sufficiently large. Denote

$$w = \sum_{l=1}^{\infty} c_m^{-1}c_{m+l}t^{q^l}, \quad c' = c_m(t+w).$$

The series $(t+w)^{-1} = \sum\limits_{n=0}^{\infty}(-1)^n w^n$ converges in the standard non-Archimedean topology of formal power series (see [83], Sect. 19.7) because the formal power series for w^n begins from the term with t^{q^n}; recall that w^n is the composition power, and t is the unit element. Moreover, $w^n = \sum\limits_{j=n}^{\infty} a_j^{(n)}t^{q^j}$ where $\left|a_j^{(n)}\right| \le C_1^{q^j-1}$ for all j, with the same constant independent of n. Using the ultra-metric inequality we find that the coefficients of the formal power series $(t+w)^{-1} = \sum\limits_{j=0}^{\infty} a_j t^{q^j}$ (each of them is, up to a sign, a finite sum of the coefficients $a_j^{(n)}$) satisfy the same estimate. Therefore $(c')^{-1} \in \mathcal{R}_K$. ■

The skew field of fractions $\mathcal{A}_K$ can be imbedded into wider skew fields where operations are more explicit. Let K_{perf} be the perfect closure of the field K. Denote by $\mathcal{A}_{K_{\text{perf}}}^{\infty}$ the composition ring of $\mathbb{F}_q$-linear formal Laurent series $a = \sum\limits_{k=m}^{\infty} a_k t^{q^k}$, $m \in \mathbb{Z}$, $a_k \in K_{\text{perf}}$, $a_m \ne 0$ (if $a \ne 0$). Since τ is an automorphism of K_{perf}, $\mathcal{A}_{K_{\text{perf}}}^{\infty}$ is a special case of the well-known ring of twisted Laurent series [83]. Therefore $\mathcal{A}_{K_{\text{perf}}}^{\infty}$ is a skew field.

Let $\mathcal{A}_{K_{\text{perf}}}$ be a subring of $\mathcal{A}_{K_{\text{perf}}}^{\infty}$ consisting of formal series with $|a_k| \le A^{q^k}$ for all $k \ge 0$. Just as in the proof of Proposition 1.4, we show that $\mathcal{A}_{K_{\text{perf}}}$ is actually a skew field. Its elements can be written in the form $t^{q^{-m}} \circ c$ where c is an invertible element of the ring $\mathcal{R}_{K_{\text{perf}}} \in \mathcal{A}_{K_{\text{perf}}}$ of formal power series $\sum\limits_{k=0}^{\infty} a_k t^{q^k}$. In contrast to the case of the skew field $\mathcal{A}_K$, in $\mathcal{A}_{K_{\text{perf}}}$ the multiplication of $t^{q^{-m}}$ by c is indeed the composition of (locally defined) functions, so that $\mathcal{A}_{K_{\text{perf}}}$ consists of fractional power series understood in the classical sense.

Of course, $\mathcal{A}_{K_{\text{perf}}}$ can be extended further, by considering $\overline{K}$ or $\overline{K}_c$ instead of K_{perf}. The above reasoning carries over to these cases (we can also consider the ring $\mathcal{R}_{\overline{K}_c}$ of locally convergent $\mathbb{F}_q$-linear power series as the initial ring). In each of them the presence of a fractional composition factor $t^{q^{-m}}$ is an $\mathbb{F}_q$-linear counterpart of a pole of order m. Thus we may see the above skew fields as consisting of functions of meromorphic type.

1.1.3. Orthonormal bases. In the sequel we will need some elementary notions from non-Archimedean functional analysis. The reader can consult [93, 100, 82] for further information on this rich and well-developed subject.

Let E be a vector space over K (similarly one can deal with any non-Archimedean local field). A norm on E is a map $u \mapsto \|u\|$ from E to $\mathbb{R}$, such that for $u, v \in E$, $\lambda \in K$,

$$\|u\| \geq 0, \quad \|\lambda u\| = |\lambda| \cdot \|u\|, \quad \|u + v\| \leq \max(\|u\|, \|v\|),$$

and $\|u\| = 0$ if and only if $u = 0$. If E is complete as a metric space with the metric $(u, v) \mapsto \|u - v\|$, then the space E is called *a Banach space* over K.

In this book we will often deal with the Banach space $C(O, K)$ of all K-valued continuous functions on the ring of integers $O = \{z \in K : \ |z| \leq 1\}$, equipped with the norm

$$\|u\| = \sup_{s \in O} |u(s)|,$$

and its subspace $C_0(O, K)$ consisting of $\mathbb{F}_q$-linear functions.

A sequence $\{f_0, f_1, f_2, \dots\}$ in a Banach space E over K is called *an orthonormal basis*, if each $u \in E$ can be represented as a convergent series

$$u = \sum_{n=0}^{\infty} c_n f_n$$

where $c_n \in K$, $c_n \to 0$, and

$$\|u\| = \max_{n \geq 0} |c_n|.$$

The coefficients in such a representation are determined in a unique way.

Denote by $P \subset K$ *the prime ideal*

$$P = xO = \{z \in K : \ |z| < 1\}.$$

Then *the residue field* O/P is isomorphic to $\mathbb{F}_q$. If $E_0 = \{u \in E : \ \|u\| \leq 1\}$, then the *residual space* (or *the reduction*) $\overline{E} = E_0/PE_0$ is a vector space over the residue field O/P. In an obvious way, any element from E_0 possesses a reduction from $\overline{E}$.

Denote by $\|E\|$ and $|K|$ the sets of values of the norm $\|u\|$, $u \in E$, and of the absolute value $|z|, z \in K$, respectively.

Proposition 1.5 *Suppose that $\|E\| = |K|$. A sequence $\{f_n\} \subset E$ is an orthonormal basis in E, if and only if all the vectors f_n lie in E_0, and their reductions $\overline{f_n} \in \overline{E}$ form an $\mathbb{F}_q$-basis of $\overline{E}$ in the algebraic sense, that is the sequence $\{\overline{f_n}\}$ is linearly independent, and every element from $\overline{E}$ is a finite linear combination of elements of this sequence.*

Proof. The necessity is evident. Let us prove the sufficiency. If $u \in E_0$, denote by $\overline{u}$ its image in $\overline{E}$. Then $\overline{u}$ can be written as a finite linear combination, $\overline{u} = \sum \xi_i \overline{f}_i$, $\xi_i \in \mathbb{F}_q$. Considering ξ_i as classes from O/P, we can take their representatives $z_i^{(1)} \in O$. Then $u - \sum z_i^{(1)} f_i \in PE_0$, so that

$$u = \sum z_i^{(1)} f_i + xz^{(1)}, \quad z^{(1)} \in E_0.$$

Repeating the above procedure for $z^{(1)}$ and iterating we come to a representation

$$u = \sum z_i f_i, \quad z_i \in O, \ z_i \to 0.$$

If $\|u\| = 1$, then necessarily $\sup\limits_i |z_i| = 1$ (otherwise $|z_i| \le q^{-1}$ for all i, so that $\|u\| < 1$). Since $\|E\| = |K|$, for any $u \in E$ we have $\|u\| = q^n$, $n \in \mathbb{Z}$, so that $\|x^n u\| = 1$, $x^n u = \sum\limits_{i=0}^{\infty} x^n z_i f_i$, and we have seen that $\sup\limits_i |x^n z_i| = 1$, whence $\sup\limits_i |z_i| = q^n = \|u\|$. ■

In fact, every separable Banach space E over the field K, such that $\|E\| = |K|$, possesses an orthonormal basis (the Monna-Fleischer theorem, see [93, 90, 98]). In this book we are interested mostly in the explicit construction of orthonormal bases for some function spaces.

Let E_1 and E_2 be Banach spaces over K with the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. The *topological tensor product* $E_1 \hat{\otimes} E_2$ is the completion of the algebraic tensor product the $E_1 \otimes_K E_2$ with respect to the norm

$$\|u\|_\otimes = \inf \left\{ \max_{1 \le i \le r} \|v_i\|_1 \cdot \|w_i\|_2 : \ u = \sum_{i=1}^r v_i \otimes w_i, \ v_i \in E_1, \ w_i \in E_2 \right\} \tag{1.7}$$

where the infimum is taken over all possible representations $u = \sum\limits_{i=1}^r v_i \otimes w_i$,