4

Lie algebroids: Algebraic constructions

This chapter is concerned with the general concept of morphism of Lie algebroids. It is unusual for the definition of a concept of morphism to require much attention. However the bracket of a Lie algebroid is defined not on the vector bundle but on its module of sections, and a morphism of vector bundles does not generally induce a map of sections. We handle this problem by considering first the case in which the target Lie algebroid may be pulled back across the base map; when this is possible, the definition of morphism presents itself naturally, and may then be reformulated so as to be viable when the pullback does not exist.

This procedure requires that pullbacks of Lie algebroids be defined before a concept of morphism can be formulated. The pullback notion for Lie groupoids of §2.3 has a clear analogue for Lie algebroids, and we proceed from this, justifying it retrospectively in 4.3.6 once the general notion of morphism has been defined.

The key to this chapter, as with Chapter 2, is the fact that the principal algebraic constructions for Lie algebroids may be characterized in terms of morphisms of specific types. This makes it possible to bypass a considerable number of explicit calculations. Rather than explicitly differentiating the algebraic constructions of Chapter 2, we proceed by analogy — based on diagrams and some very basic categorical principles — to define corresponding notions for abstract Lie algebroids. The characterization of these notions in terms of classes of morphisms, together with the functoriality of the process of taking the Lie algebroid, then gives immediately that differentiation — the Lie functor — maps the Lie groupoid constructions to their Lie algebroid analogues.

The chapter begins with infinitesimal actions of Lie algebroids. This case can be treated without a general notion of morphism and indeed provides the best basis on which to introduce the general notion. §4.2 is a rapid treatment of direct products and pullbacks preparatory to the definition of general morphisms in §4.3. The remaining sections treat general quotients, generalized actions and the most general notion of semidirect product.

4.1 Actions of Lie algebroids

The first step is to extend the definition of an infinitesimal action of a Lie algebra on a manifold (as in 3.3.7) to Lie algebroids.

Definition 4.1.1 Let A be a Lie algebroid on M, and let $f\colon M' \to M$ be a smooth map. Then an *infinitesimal action of* A *on* f, or *on* M', is a map $X \mapsto X^\dagger$, $\Gamma A \to \mathfrak{X}(M')$ satisfying the four conditions:

$$(X+Y)^\dagger = X^\dagger + Y^\dagger \quad (1)$$

$$(uX)^\dagger = (u\circ f)X^\dagger \quad (2)$$

$$[X,Y]^\dagger = [X^\dagger, Y^\dagger] \quad (3)$$

$$X^\dagger \overset{f}{\sim} a(X) \quad (4)$$

for $X, Y \in \Gamma A$, $u \in C^\infty(M)$. □ □

The last condition is that $X^\dagger$, a vector field on M', projects under $f\colon M' \to M$ to $a(X)$, a vector field on M. Equivalently, $X^\dagger(u\circ f) = a(X)\circ f$ for all $u \in C^\infty(M)$.

We drop the word 'infinitesimal' except when emphasis is needed.

This definition is justified by two observations: firstly, an action of a Lie groupoid G, as defined in 1.6.1, induces an action of AG in this sense (see 4.1.6); and, secondly, there is an abstract version of the correspondence of §1.6 between groupoid actions and action morphisms. We establish the second point first.

Suppose given a Lie algebroid A on M, a smooth map $f\colon M' \to M$, and an infinitesimal action $X \mapsto X^\dagger$ as in 4.1.1. We will define a Lie algebroid structure on the pullback vector bundle $A' = f^!A$, regarding the sections of $f^!A$ as sums $\sum u'_i \otimes X_i$ where $u'_i \in C^\infty(M')$, $X_i \in \Gamma A$ (see §2.1). Here the tensor product is over $C^\infty(M)$ and the isomorphism of $C^\infty(M')$–modules $C^\infty(M') \otimes \Gamma A \cong \Gamma(f^!A)$ is $u' \otimes X \mapsto (u'\circ f)X^!$, where $X^!$ is the pullback of the section X.

Proposition 4.1.2 *Let A be a Lie algebroid on M, and $f\colon M' \to M$ a* *smooth map. Let* $X \mapsto X^\dagger$ *be an infinitesimal action as in* 4.1.1. *Then an anchor and a bracket structure are defined on* $f^!A$ *by*

$$a'\left(\sum u'_i \otimes X_i\right) = \sum u'_i X_i^\dagger \tag{5}$$

$$\left[\sum u'_i \otimes X_i, \sum v'_j \otimes Y_j\right] = \sum u'_i v'_j \otimes [X_i, Y_j] + \sum u'_i X_i^\dagger(v'_j) \otimes Y_j - \sum v'_j Y_j^\dagger(u'_i) \otimes X_i, \tag{6}$$

where $u'_i, v'_j \in C^\infty(M')$, $X_i, Y_j \in \Gamma A$. *With this structure,* $f^!A$ *is a Lie algebroid on* M'.

Proof Note first that if $w \in C^\infty(M)$ then $a'(u' \otimes (wX)) = u'(w \circ f)X^\dagger = a'(u'(w \circ f) \otimes X)$, using (2), so a' is well defined. It is $C^\infty(M')$–linear and so defines a vector bundle morphism $f^!A \to TM'$.

Now fix $u'_i \in C^\infty(M')$ and $X_i \in \Gamma A$. Define a map

$$E\colon\ C^\infty(M') \times \Gamma A \to C^\infty(M') \otimes \Gamma A$$

by

$$E(v', Y) = \sum u'_i v' \otimes [X_i, Y] + \sum u'_i X_i^\dagger(v') \otimes Y - \sum v' Y^\dagger(u'_i) \otimes X_i.$$

It is easy to see that $E(v', wY) = E(v'(w \circ f), Y)$ for $w \in C^\infty(M)$, and so E defines a $C^\infty(M')$–linear map $\widetilde{E}\colon\ C^\infty(M') \otimes \Gamma A \to C^\infty(M') \otimes \Gamma A$ by

$$\widetilde{E}\left(\sum v'_j \otimes Y_j\right) = \sum E(v'_j, Y_j).$$

Thus the bracket in (6) is well defined in its second variable. A similar argument applies to the first.

To verify the Leibniz condition it suffices, since both sides are $\mathbb{R}$–bilinear, to consider elements $u' \otimes X$. Then

$$\begin{aligned}[u' \otimes X, w'(v' \otimes Y)] &= [u' \otimes X, (v'w') \otimes Y] \\ &= u'v'w' \otimes [X, Y] + u'X^\dagger(v'w') \otimes Y - v'w'Y^\dagger(u') \otimes X \\ &= w'[u' \otimes X, v' \otimes Y] + u'v'X^\dagger(w') \otimes Y \\ &= w'[u' \otimes X, v' \otimes Y] + a'(u' \otimes X)(w').(v' \otimes Y),\end{aligned} \tag{7}$$

as required. The Jacobi identity and bracket–preservation for a' are proved in the same way. $\square$

We denote $f^!A$ with this structure by $A \lessdot f$ or $A \lessdot M'$; it is the *action Lie algebroid* or *transformation Lie algebroid* corresponding to $X \mapsto X^\dagger$.

Example 4.1.3 Let A be any Lie algebroid on a connected base M and let $f\colon \widetilde{M} \to M$ be any covering. Then there is a canonical action of A on f in which $X^\dagger$ is the π–invariant lift of $a(X)$ to $\widetilde{M}$, where π is the group of the covering.

For $A = TM$ the canonical map $T(\widetilde{M}) \to TM \lessdot f$ is an isomorphism of Lie algebroids. ⊠

Examples 4.1.4 Any Lie groupoid $G \rightrightarrows M$ acts on its target projection by the groupoid multiplication. The action Lie algebroid $AG \lessdot \beta$ is canonically isomorphic to the distribution $T^\alpha G$ under the map $\mathscr{R}$ of 3.5.3 which sends $1 \otimes X \in \Gamma(AG \lessdot \beta)$ to $\overrightarrow{X}$. Compare 1.6.11.

In particular, for any Lie group G, let $\mathfrak{g}$ act on the manifold G by $X^\dagger = \overrightarrow{X}$. Then the anchor $\mathfrak{g} \lessdot G \to TG$ of the action Lie algebroid is an isomorphism onto the standard Lie algebroid TG. This is equivalent to the fact that if, for any $X \in \mathfrak{X}(G)$, the function $X^R\colon G \to \mathfrak{g}$ is defined by $X^R(g) = T(R_{g^{-1}})(X(g))$, then

$$[X, Y]^R = \mathfrak{L}_X(Y^R) - \mathfrak{L}_Y(X^R) + [X^R, Y^R]^\bullet$$

where $[\ ,\ ]^\bullet$ is the pointwise bracket.

Similarly, for any principal bundle $P(M, G)$, the standard Lie algebroid TP is isomorphic to the action Lie algebroid $\frac{TP}{G} \lessdot P$. ⊠

We now consider a Lie algebroid version of 1.6.16. Let A and A' be Lie algebroids on bases M and M', and suppose that

$$\begin{array}{ccc} A' & \xrightarrow{\ \varphi\ } & A \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ M' & \xrightarrow[\ f\ ]{} & M \end{array} \qquad (8)$$

is a pullback of vector bundles. Thus the vector bundle A' is isomorphic to the pullback vector bundle $f^!A \to M'$ under the map $\varphi^!\colon A' \to f^!A,\ X' \mapsto (q'X', \varphi X')$. For $X \in \Gamma A$, denote by $X^!$ the unique section of A' with $\varphi \circ X^! = X \circ f$. If M is singleton, then the sections $X^!$ are the constant sections of the trivial(izable) bundle A'. Generalizing that case, let us impose on (8) the two conditions that

$$T(f)\circ a' = a\circ\varphi, \tag{9}$$

$$[X^!, Y^!] = [X,Y]^! \quad \text{for all } X,Y\in\Gamma A. \tag{10}$$

We will see shortly that, in this case, these are precisely the conditions that (φ, f) is a Lie algebroid morphism.

In the case where M was singleton, the original action could be recovered from the anchor. In the present situation we therefore define a map $\Gamma A\to\mathfrak{X}(M')$ by $X\mapsto X^\dagger = a'(X^!)$. It is straightforward to verify that conditions (1) – (4) hold and we have the following result.

Proposition 4.1.5 *Let A and A' be Lie algebroids on M and M' and let $\varphi\colon A'\to A$, $f\colon M'\to M$ be a pullback of vector bundles satisfying* (9) *and* (10). *Then, with the above notation, $X^\dagger = a'(X^!)$ defines an infinitesimal action of A on f, and $\varphi^!\colon A'\to A\lessdot f$ is an isomorphism of Lie algebroids over M'.*

There is thus established a bijective correspondence between actions of A on the map $f\colon M'\to M$ and maps $\varphi\colon A'\to A$, $f\colon M'\to M$, which are pullback morphisms of vector bundles, which commute with the anchors on A' and A, and which satisfy (10). In §4.3 such maps (φ, f) will re–emerge as *action morphisms* of Lie algebroids.

Consider now a Lie groupoid action $G*M'\to M'$ of a Lie groupoid $G\rightrightarrows M$ on a smooth map $f\colon M'\to M$. Then, as in §1.6, the canonical map $F\colon G\lessdot f\to G$, $(g,m')\mapsto g$, is a morphism of Lie groupoids and so induces, by 3.5.10, a vector bundle morphism AF from $A(G\lessdot f)$ to AG which satisfies the two conditions (9) and (10). Applying 4.1.5, there is an infinitesimal action of AG on f given for $X\in\Gamma AG, m'\in M'$ by

$$X^\dagger(m') = T_{1_{f(m')}}(g\mapsto gm')(X(f(m'))) \tag{11}$$

and we have the following theorem.

Theorem 4.1.6 *Let $G\rightrightarrows M$ be a Lie groupoid acting on a smooth map $f\colon M'\to M$. Let $F\colon G\lessdot f\to G$ be the corresponding action morphism. Then $(AF)^!\colon A(G\lessdot f)\to(AG)\lessdot f$ is an isomorphism of Lie algebroids, where the action of AG on f is given by* (11).

When a Lie groupoid acts linearly on a vector bundle there are now two induced Lie algebroid phenomena: the induced representation and the infinitesimal action. These are related in the natural way:

Proposition 4.1.7 *Let ρ be a linear action of a Lie groupoid $G \rightrightarrows M$ on a vector bundle E. Then the induced representation $\rho_*\colon AG \to \mathfrak{D}(E)$ and the infinitesimal action $X \mapsto X^\dagger$, $\Gamma AG \to \mathfrak{X}(E)$ are related by*

$$\langle \varphi, \rho_X(\mu)\rangle = a(X)\langle \varphi, \mu\rangle - X^\dagger(\ell_\varphi) \circ \mu$$

for $\varphi \in \Gamma E^*$, $\mu \in \Gamma E$.

Proof This is simply the statement that, for all $X \in \Gamma AG$, the derivation corresponding to the linear vector field $(X^\dagger, a(X))$ is ρ_X (see 3.4.5). □

The three equations of 3.4.5 now also apply to ρ_X and $X^\dagger$. In particular, for $X \in \Gamma AG$, $\mu \in \Gamma E$,

$$X^\dagger(\mu(m)) = T(\mu)(a(X)(m)) - \rho_X(\mu)^\uparrow(\mu(m)), \tag{12}$$

$$\rho_X(\mu)^\uparrow = [X^\dagger, \mu^\uparrow]. \tag{13}$$

Of course, these formulas also give a correspondence between representations $\rho\colon A \to \mathfrak{D}(E)$ of abstract Lie algebroids and infinitesimal actions $\Gamma A \to \mathfrak{X}(E)$ by linear vector fields.

Example 4.1.8 Let $\mathfrak{g}$ be a Lie algebra and $E \to M$ a vector bundle. A *derivative representation* of $\mathfrak{g}$ on E is a Lie algebra morphism $\kappa\colon \mathfrak{g} \to \Gamma\,\mathfrak{D}(E)$. Composing with the anchor $\mathfrak{D}(E) \to TM$, a derivative representation induces an infinitesimal action of $\mathfrak{g}$ on M. If κ is now lifted to $\mathfrak{g} \lessdot M$, we have a representation (in the sense of 3.3.13) of the action Lie algebroid on E. Conversely, any infinitesimal action of $\mathfrak{g}$ on M and a representation $\mathfrak{g} \lessdot M \to \mathfrak{D}(E)$ define a derivative representation by restricting to constant sections. ⊠

Use of connections

There is an alternative form for the bracket formula (6), obtained by using an auxiliary connection in the vector bundle underlying A.

Start with an arbitrary Lie algebroid A on M and a Koszul connection ∇ in A (see §5.2). The usual notion of torsion for connections in tangent bundles extends to this case, and we define T_∇, the *torsion of* ∇ for $X, Y \in \Gamma A$ by

$$T_\nabla(X,Y) = \nabla_{aX}(Y) - \nabla_{aY}(X) - [X,Y]. \tag{14}$$

As a map $\Gamma A \times \Gamma A \to \Gamma A$ this is bilinear over $C^\infty(M)$ and hence defines $T_\nabla\colon A \oplus A \to A$ which is alternating.

Let $f\colon M' \to M$ be an arbitrary smooth map and denote by $\overline{\nabla}$ the pullback connection in $f^! A$. Thus

$$\overline{\nabla}_{V'}\left(\sum u'_i \otimes X_i\right) = \sum u'_i \otimes \nabla_{T(f)(V')}(X_i) + \sum V'(u'_i) \otimes X_i$$

where $V' \in TM'$, $u'_i \in C^\infty(M')$, $X_i \in \Gamma A$. A priori, we do not have a Lie algebroid structure in $f^! A$, so there is no concept of torsion for $\overline{\nabla}$. However we pull back T_∇ to $f^! A$, obtaining $\overline{T}_\nabla\colon f^! A \oplus f^! A \to f^! A$. The following is now a simple check.

Proposition 4.1.9 *Let A act on $f\colon M' \to M$ by $X \mapsto X^\dagger$. Then the bracket* (6) *in* $\Gamma(A \lessdot f)$ *is*

$$[C,D] = \overline{\nabla}_{\overline{a}(C)}(D) - \overline{\nabla}_{\overline{a}(D)}(C) - \overline{T}_\nabla(C,D) \tag{15}$$

where $C, D \in \Gamma(A \lessdot f)$, and $\overline{a}$ is the anchor $A \lessdot f \to TM'$. The Lie algebroid torsion of $\overline{\nabla}$ coincides with $\overline{T}_\nabla$.

The advantage of this formulation is that, once ∇ is chosen, each term on the RHS is well defined and there is no further need to deal with the representation of elements of $\Gamma(f^! A)$ as tensor products.

Example 4.1.10 Let G be a Lie group and H a closed subgroup. Let G act on $M = G/H$ in the standard way, and let $\mathfrak{g} \lessdot M$ be the action Lie algebroid. The vector bundle $\mathfrak{g} \to \{*\}$ has a unique (trivial) connection, the torsion of which, according to (14), is $T(X,Y) = -[X,Y]$ for $X, Y \in \mathfrak{g}$. The pullback connection is the standard flat connection ∇^0 in $M \times \mathfrak{g}$ and we have the bracket formula for $\mathfrak{g} \lessdot M$

$$[V,W] = V^\dagger(W) - W^\dagger(V) + [V,W]^\bullet,$$

as in (15) of Chapter 3, with the Lie algebroid torsion being the negative of the pointwise bracket.

When H is the trivial subgroup, the Lie algebroid torsion reduces to the usual concept and (15) is the usual formula for the torsion of the right–invariant flat connection in G. ⊠

4.2 Direct products and pullbacks of Lie algebroids

Consider Lie algebroids $A^1 \to M^1$ and $A^2 \to M^2$. We want to define a Lie algebroid structure on $A^1 \times A^2 \to M^1 \times M^2$ in such a way that the bracket of two sections which both come from A^1 is determined by their bracket in $\Gamma(A^1)$, likewise for A^2, and such that the bracket of two sections, one from A^1 and one from A^2, is zero. More care needs to be taken with the formulation of these requirements than is the case with Lie algebras — by 'a section which comes from A^1' we must mean not merely a section which takes values in A^1, but one which also does not depend on M^2. However we will see that these requirements, together with the Leibniz condition, determine a Lie algebroid structure on $A^1 \times A^2$.

Denote the projections from $M^1 \times M^2$ to M^1, M^2 by $\mathrm{pr}_1, \mathrm{pr}_2$. The product vector bundle $A^1 \times A^2 \to M^1 \times M^2$ can be regarded as the Whitney sum over $M^1 \times M^2$ of the pullback vector bundles $\mathrm{pr}_1^! A^1$ and $\mathrm{pr}_2^! A^2$. Sections of $\mathrm{pr}_1^! A^1$ are of the form $\sum u_i \otimes X_i^1$ where $u_i \in C^\infty(M^1 \times M^2)$ and $X_i^1 \in \Gamma(A^1)$. Likewise, we write sections of $\mathrm{pr}_2^! A^2$ in the form $\sum u_j' \otimes X_j^2$ where $u_j' \in C^\infty(M^1 \times M^2)$ and $X_j^2 \in \Gamma(A^2)$.

The tangent bundle $T(M^1 \times M^2)$ may in the same way be regarded as the Whitney sum $\mathrm{pr}_1^!(TM^1) \oplus \mathrm{pr}_2^!(TM^2)$. We define the anchor $a\colon A^1 \times A^2 \to T(M^1 \times M^2)$ by using these descriptions and extending a_1 and a_2 linearly; thus

$$a\left(\sum (u_i \otimes X_i^1) \oplus \sum (u_j' \otimes X_j^2)\right) = \sum (u_i \otimes a_1(X_i^1)) \oplus \sum (u_j' \otimes a_2(X_j^2)).$$

Now we impose the conditions

$$\begin{aligned} [1 \otimes X^1, 1 \otimes Y^1] &= 1 \otimes [X^1, Y^1], \qquad [1 \otimes X^1, 1 \otimes Y^2] = 0, \\ [1 \otimes X^2, 1 \otimes Y^2] &= 1 \otimes [X^2, Y^2], \qquad [1 \otimes X^2, 1 \otimes Y^1] = 0, \end{aligned} \tag{16}$$

for $X^1, Y^1 \in \Gamma(A^1)$, $X^2, Y^2 \in \Gamma(A^2)$. It follows that for

$$X = \sum (u_i \otimes X_i^1) \oplus \sum (u_j' \otimes X_j^2) \quad \text{and} \quad Y = \sum (v_k \otimes Y_k^1) \oplus \sum (v_\ell' \otimes Y_\ell^2),$$

we have, using the Leibniz condition,

$$\begin{aligned} [X, Y] = \Big\{ &\sum u_i v_k \otimes [X_i^1, Y_k^1] + \sum u_i a_1(X_i^1)(v_k) \otimes Y_k^1 \\ &- \sum v_k a_1(Y_k^1)(u_i) \otimes X_i^1 \Big\} \\ \oplus \Big\{ &\sum u_j' v_\ell' \otimes [X_j^2, Y_\ell^2] + \sum u_j' a_2(X_j^2)(v_\ell') \otimes Y_\ell^2 \\ &- \sum v_\ell' a_2(Y_\ell^2)(u_j') \otimes X_j^2 \Big\} \end{aligned} \tag{17}$$

where we write $a_1(X^1)(v)$ as shorthand for the action of $a_1(X^1)^!$ on $v \in C^\infty(M^1 \times M^2)$.

It is now a straightforward exercise to verify that this bracket and anchor are well defined and make $A^1 \times A^2$ a Lie algebroid over $M^1 \times M^2$, which we call the *direct product Lie algebroid.* We postpone a more formal statement to §4.3 since we do not yet have a concept of morphism. Note however, that if $A^1 = TM^1$, $A^2 = TM^2$, then the product Lie algebroid is canonically isomorphic (over $M^1 \times M^2$) to $T(M^1 \times M^2)$.

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We now turn to pullbacks of Lie algebroids over smooth maps. Given a Lie algebroid A on M and a smooth map $f\colon M' \to M$, the two maps in the diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow a \\ TM' & \xrightarrow[T(f)]{} & TM \end{array} \qquad (18)$$

should be included in any definition of Lie algebroid morphism. Suppose that the vector bundle pullback $TM' \oplus_{TM} A$ of (18) exists. Thus its elements are $x' \oplus X$ where $x' \in TM'$ and $X \in A$ are subject to $T(f)(x') = a(X)$. When dealing with sections, it is convenient to regard $TM' \oplus_{TM} A$ as the pullback of $T(f)^!\colon TM' \to f^!TM$ and $f^!(a)\colon f^!(A) \to f^!(TM)$ in the category of vector bundles over M'. Thus sections of $TM' \oplus_{TM} A$ are expressions

$$x' \oplus \left(\sum u_i' \otimes X_i\right), \qquad x' \in \mathfrak{X}(M'),\ u_i' \in C^\infty(M'),\ X_i \in \Gamma A,$$

such that $T(f)(x')(m') = \sum u_i'(m')a(X_i(f(m')))$ for all $m' \in M'$.

On the basis that a pullback should be a subobject of the direct product, define an anchor a' by

$$a'\left(x' \oplus \left(\sum u_i' \otimes X_i\right)\right) = x', \qquad (19)$$

and a bracket by

$$\left[x' \oplus \left(\sum u_i' \otimes X_i\right), y' \oplus \left(\sum v_j' \otimes Y_j\right)\right] = \\ [x', y'] \oplus \Big(\sum u_i' v_j' \otimes [X_i, Y_j] + \\ \sum a'(x')(v_j') \otimes Y_j - \sum a'(y')(u_i') \otimes X_i\Big). \quad (20)$$

Again, it is a straightforward task to verify that, with this structure, $TM' \oplus_{TM} A$ is a Lie algebroid on M'. We postpone a formal statement to 4.3.6. With this structure, we denote $TM' \oplus_{TM} A$ by $f^{!!}A$ and call it the *pullback Lie algebroid of* A *over* f.

The pullback Lie algebroid always exists if A is transitive, or if f is a surjective submersion.

Example 4.2.1 Taking $A = \mathfrak{g}$ to be a Lie algebra, the pullback of $\mathfrak{g}$ to a manifold M' is the trivial Lie algebroid of 3.3.3. ⊠

4.3 Morphisms of Lie algebroids

A general vector bundle map

$$\begin{array}{ccc} A' & \xrightarrow{\varphi} & A \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ M' & \xrightarrow[f]{} & M \end{array} \quad (21)$$

does not induce a map from the sections of A' to the sections of A, and the simple definition 3.3.1 of a morphism of Lie algebroids available in the base preserving case does not apply here.

Suppose first of all that the pullback $f^{!!}A$ of the target exists. Then, by analogy with the decomposition of Lie groupoid morphisms in §2.3, we could define φ to be a morphism if

$$a \circ \varphi = T(f) \circ a', \quad (22)$$

and if the map

$$\varphi^{!!}\colon A' \to f^{!!}A, \qquad X' \mapsto a'(X') \oplus \varphi(X'), \quad (23)$$

is a morphism (over M') into $f^{!!}A$. By (19), (20) this last condition is