Lie Algebras and Root Systems

R.W. Carter

Contents
Lie Algebras and Root Systems

Preface

The following notes on Lie Algebras and Root Systems follow fairly closely the lectures I gave on this subject at the Lancaster meeting, although more detail has been included in a number of places. The aim has been to give an outline of the main ideas involved in the structure and representation theory of the simple Lie algebras over $\mathbb{C}$, and the construction of the corresponding groups of Lie type over an arbitrary field.

It has not been possible to give all the proofs in detail, and so interested readers are encouraged to consult books in which more complete information is given. The following books are particularly recommended.

J. E. Humphreys, Introduction to Lie Algebras and Representation Theory, Graduate Texts in Mathematics 9 (1972) Springer.

N. Jacobson, Lie Algebras. Interscience Publishers, J. Wiley, New York (1962).

R. W. Carter, Simple Groups of Lie Type, Wiley Classics Library Edition (1989), J. Wiley, New York.

Cambridge University Press
978-0-521-49922-4 - Lectures on Lie Groups and Lie Algebras
Roger Carter, Graeme Segal and Ian Macdonald
Excerpt
More information

1
Introduction to Lie algebras

1.1 Basic concepts

A Lie algebra is a vector space g over a field F on which a multiplication

$$\begin{aligned} g \times g &\to g \\ x, y &\to [xy] \end{aligned}$$

is defined satisfying the axioms:

(i) $[xy]$ is linear in x and in y.
(ii) $[xx] = 0$ for all $x \in g$.
(iii) $[[xy]z] + [[yz]x] + [[zx]y] = 0$ for all $x, y, z \in g$.

Property (iii) is called the Jacobi identity.

We note that the multiplication is not associative, i.e., it is not true in general that $[[xy]z] = [x[yz]]$. It is therefore essential to include the Lie brackets in products of elements.

For any pair of elements $x, y \in g$ we have

$$[x + y, x + y] = [xx] + [xy] + [yx] + [yy].$$

We also know that

$$[xx] = 0, \quad [yy] = 0, \quad [x + y, x + y] = 0.$$

It follows that $[yx] = -[xy]$ for all $x, y \in g$. Thus multiplication in a Lie algebra is anticommutative.

Lie algebras can be obtained from associative algebras by the following method. Let A be an associative algebra, i.e., a vector space with a bilinear associative multiplication xy. Then we may obtain a Lie algebra $[A]$ by redefining the multiplication on A. We define $[xy] = xy - yx$. It is clear that $[xy]$ is linear in x and in y and that $[xx] = 0$. We also have

$$\begin{aligned} [[xy]z] &= (xy - yx)z - z(xy - yx) \\ &= xyz - yxz - zxy + zyx. \end{aligned}$$

It follows that

$$\begin{aligned} & [[xy]z] + [[yz]x] + [[zx]y] \\ = \; & xyz - yxz - zxy + zyx \\ & + yzx - zyx - xyz + xzy \\ & + zxy - xzy - yzx + yxz \\ = \; & 0, \end{aligned}$$

so that the Jacobi identity is satisfied.

Let g_1, g_2 be Lie algebras over F. A homomorphism of Lie algebras is a linear map $\theta : g_1 \to g_2$ such that $\theta[xy] = [\theta x, \theta y]$ for all $x, y \in g_1$.

θ is an isomorphism of Lie algebras if θ is a bijective homomorphism.

Let g be a Lie algebra and h, k be subspaces of g. We define the product $[hk]$ to be the subspace spanned by all products $[xy]$ for $x \in h$, $y \in k$. Each element of $[hk]$ is thus a finite sum $x_1 y_1 + \cdots + x_r y_r$ with $x_i \in h$, $y_i \in k$. We note that $[hk] = [kh]$, i.e., multiplication of subspaces is commutative. This follows from the fact that multiplication of elements is anticommutative. So if $x \in h$, $y \in k$ we have $[yx] = -[xy] \in [hk]$.

A subalgebra of g is a subspace h of g such that $[hh] \subset h$.

An ideal of g is a subspace h of g such that $[hg] \subset h$.

We observe that, since $[hg] = [gh]$, there is no distinction in the theory of Lie algebras between left ideals and right ideals. Every ideal is two-sided.

Now let h be an ideal of the Lie algebra g. Let g/h be the vector space of cosets $h + x$ for $x \in g$. $h + x$ consists of all elements of form $y + x$ for $y \in h$. We claim that g/h can be made into a Lie algebra, the factor algebra of g with respect to h, by introducing the Lie multiplication

$$[h + x, h + y] = h + [xy].$$

We must take care to check that this operation is well defined, i.e., that if $h + x = h + x'$ and $h + y = h + y'$ then $h + [xy] = h + [x'y']$. This follows from the fact that h is an ideal of g. We have

$$x' = a + x, \quad y' = b + y \quad \text{for } a, b \in h.$$

Thus

$$[x'y'] = [ab] + [ay] + [xb] + [xy] \in h + [xy]$$

since $[ab], [ay], [xb]$ all lie in h. This gives $h + [x'y'] = h + [xy]$ as required.

There is a natural homomorphism $g \xrightarrow{\theta} g/h$ relating a Lie algebra with a factor algebra. θ is defined by $\theta(x) = h + x$. Conversely given any homomorphism $\theta : g_1 \to g_2$ of Lie algebras which is surjective, the kernel k of θ is an ideal of g_1 and the factor algebra g_1/k is isomorphic to g_2.

The set of all $n \times n$ matrices over the field F can be made into a Lie algebra under the Lie multiplication $[A, B] = AB - BA$. This Lie algebra is called $gl_n(F)$, the general linear Lie algebra of degree n over the field F.

1.2 Representations and modules

Let g be a Lie algebra over F. A representation of g is a homomorphism

$$\rho : g \to gl_n(F)$$

for some n. Two representations ρ, ρ' of g of degree n are called equivalent if there is a non-singular $n \times n$ matrix T over F such that

$$\rho'(x) = T^{-1}\rho(x)T, \quad \text{for all } x \in g.$$

There is a close connection between representations of g and g-modules.

A left g-module is a vector space V over F with a multiplication

$$\begin{array}{ccc} g \times V & \to & V \\ x, v & \to & xv \end{array}$$

satisfying the axioms

(i) xv is linear in x and in v
(ii) $[xy]v = x(yv) - y(xv)$ for all $x, y \in g$, $v \in V$.

Every finite dimensional g-module gives a representation of g, as follows. Choose a basis $e_1, \ldots, e_n$ of V. Then xe_j is a linear combination of $e_1, \ldots, e_n$. Let

$$xe_j = \sum_{i=1}^{n} \rho_{ij}(x)e_i.$$

Let $\rho(x)$ be the $n \times n$ matrix $(\rho_{ij}(x))$. Then we have

$$\rho[xy] = \rho(x)\rho(y) - \rho(y)\rho(x) = [\rho(x)\,\rho(y)]$$

and so the map $x \to \rho(x)$ is a representation of g.

If we choose a different basis for the g-module V we shall get an equivalent representation.

Now let U be a subspace of V and h a subspace of g. Let hU be the subspace of V spanned by all elements xu for $x \in h$, $u \in U$. U is called a submodule of V if $gU \subset U$. A g-module V is called irreducible if V has no submodules other than V and 0.

Now g is itself a g-module under the multiplication $g \times g \to g$ given by $x, y \to [xy]$. To see this we must check $[[xy]z] = [x[yz]] - [y[xz]]$ for x, y, $z \in g$. This follows from the Jacobi identity using the anticommutative law. g is called the adjoint g-module, and it gives rise to the adjoint representation of g.

1.3 Special kinds of Lie algebra

So far the theory of Lie algebras has been very analogous to the theory of rings, where one has subrings, ideals, factor rings, etc. However there is also a sense in which the theory of Lie algebras can be considered as analogous to the theory of groups, where the Lie product $[xy]$ is regarded as analogous to the commutator $x^{-1}y^{-1}xy$ of two elements in a group. This analogy motivates the following terminology.

A Lie algebra g is called abelian if $[gg] = 0$. This means that all Lie products are zero.

We shall now define a sequence of subspaces $g^1, g^2, g^3, \cdots$ of g. We define them inductively by

$$g^1 = g, \quad g^{n+1} = [g^n g].$$

Now if h, k are ideals of g so is their product $[hk]$. For let $x \in h$, $y \in k$, $z \in g$. Then we have

$$[[xy]z] = [x[yz]] + [[xz]y] \in [hk].$$

Thus the product of two ideals is an ideal. It follows that all the subspaces g^i defined above are ideals of g. Thus we also have

$$g^{n+1} = [g^n g] \subset g^n$$

and so we have a descending series

$$g = g^1 \supset g^2 \supset g^3 \supset \cdots.$$

The Lie algebra g is called nilpotent if $g^i = 0$ for some i. Every abelian Lie algebra is nilpotent.

Example. The set of all $n \times n$ matrices (a_{ij}) over F with $a_{ij} = 0$ whenever $i \geqslant j$ is a nilpotent Lie algebra under Lie multiplication $[AB] = AB - BA$.

Cambridge University Press
978-0-521-49922-4 - Lectures on Lie Groups and Lie Algebras
Roger Carter, Graeme Segal and Ian Macdonald
Excerpt
More information

We now define a different sequence of subspaces $g^{(0)}, g^{(1)}, g^{(2)}, \cdots$ of g. We again define them inductively by

$$g^{(0)} = g, \qquad g^{(n+1)} = \left[g^{(n)}g^{(n)}\right].$$

The $g^{(i)}$ are all ideals of g. Also we have

$$g^{(n+1)} = \left[g^{(n)}g^{(n)}\right] \subset g^{(n)}$$

and so we again have a descending series

$$g = g^{(0)} \supset g^{(1)} \supset g^{(2)} \supset \cdots.$$

The Lie algebra g is called soluble if $g^{(i)} = 0$ for some i.

Proposition. *Every nilpotent Lie algebra is soluble.*

Proof We show first that $[g^m g^n] \subset g^{m+n}$ for all m, n. We proceed by induction on n, the result being clear if $n = 1$. Assuming inductively that $[g^m g^n] \subset g^{m+n}$, let $x \in g^m$, $y \in g^n$, $z \in g$. Then we have

$$[x[yz]] = [[xy]z] - [[xz]y] \in g^{m+n+1}$$

by induction. Thus $[g^m g^{n+1}] \subset g^{m+n+1}$ as required.

We next observe that $g^{(n)} \subset g^{2^n}$. This is clear for $n = 0$. Assuming it inductively we have

$$g^{(n+1)} = [g^{(n)}g^{(n)}] \subset [g^{2^n}g^{2^n}] \subset g^{2^{n+1}}$$

as above. This completes the induction.

We now assume that g is nilpotent. Then $g^m = 0$ for some m. Hence there exists n with $g^{2^n} = 0$. It follows that $g^{(n)} = 0$ and so g is soluble. □

Example. The set of all $n \times n$ matrices (a_{ij}) over F with $a_{ij} = 0$ whenever $i > j$ is a soluble Lie algebra.

A Lie algebra g is called simple if g has no ideals other than g and 0.

A Lie algebra g of dimension 1 is of course simple because g has no proper subspaces at all. We have $g = Kx$ for some $x \in g$. Since $[xx] = 0$ we have $[gg] = 0$. Such a 1-dimensional Lie algebra will be called a trivial simple Lie algebra. We shall be mainly interested in non-trivial simple Lie algebras.

1.4 The Lie algebras $sl_n(\mathbb{C})$

We shall now take $F = \mathbb{C}$. Let $sl_n(\mathbb{C})$ be the set of all $n \times n$ matrices of trace 0. $sl_n(\mathbb{C})$ is an ideal of $gl_n(\mathbb{C})$. For if $A \in sl_n(\mathbb{C})$, $B \in gl_n(\mathbb{C})$ we have

$$\text{trace}[AB] = \text{trace}(AB - BA) = \text{trace}AB - \text{trace}BA = 0$$

since traceAB = traceBA for any two $n \times n$ matrices. Hence $[AB] \in sl_n(\mathbb{C})$. Thus we see that $gl_n(\mathbb{C})$ is not simple.

$sl_n(\mathbb{C})$ is, however, a non-trivial simple Lie algebra when $n \geq 2$. To see this suppose we have a non-zero ideal k and take a non-zero element in this ideal. By multiplying this element on the left or right by suitable elementary matrices E_{ij} with $i \neq j$ we may simplify its form, while remaining within the ideal k. E_{ij} is the matrix with 1 in the i, j position and 0 elsewhere. Eventually we see that k contains some elementary matrix E_{ij}, and by further multiplication we see readily that k is the whole of $sl_n(\mathbb{C})$. Thus $sl_n(\mathbb{C})$ is simple.

We shall describe certain properties of $sl_n(\mathbb{C})$ in detail, because it is typical of simple Lie algebras in general.

Let h be the set of diagonal $n \times n$ matrices of trace 0. Then h is a subalgebra of $sl_n(\mathbb{C})$ and $\dim h = n - 1$. Furthermore we have $[hh] = 0$, so h is abelian.

We recall that g may be considered as a g-module, using $[gg] \subset g$. We thus have $[hg] \subset g$ and so we may regard g as a left h-module. We may write down a decomposition of g as a direct sum of h-submodules:

$$sl_n(\mathbb{C}) = h \oplus \sum_{i \neq j} \mathbb{C}E_{ij}.$$

We note that the 1-dimensional space $\mathbb{C}E_{ij}$ is an h-submodule since, for $x \in h$, we have

$$x = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

with $\lambda_1 + \cdots + \lambda_n = 0$ and

$$[xE_{ij}] = (\lambda_i - \lambda_j)E_{ij}.$$

This h-module gives a 1-dimensional representation of h

$$x = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \longrightarrow \lambda_i - \lambda_j.$$

We note that there are $n(n-1)$ 1-dimensional representations of h arising in this way. They are called the roots of $sl_n(\mathbb{C})$ with respect to h. Let Φ be the set of roots. Φ lies in $h^* = \mathrm{Hom}(h, \mathbb{C})$, the dual space of h.

We note that if $\alpha \in \Phi$ then $-\alpha \in \Phi$ also since the map $x \to \lambda_j - \lambda_i$ is the negative of the map $x \to \lambda_i - \lambda_j$. Thus the roots are certainly not linearly independent. The roots do however span h^*. For define $\alpha_i \in \Phi$ by

$$\alpha_i(x) = \lambda_i - \lambda_{i+1}.$$

Then $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are linearly independent and form a basis of h^*. Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$. Π is called a set of fundamental roots, or simple roots. We consider the way in which the roots are expressed as linear combinations of the fundamental roots. The root $x \to \lambda_i - \lambda_j$ is equal to

$$\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1} \qquad \text{if } i < j$$

and to

$$-(\alpha_j + \alpha_{j+1} + \ldots + \alpha_{i-1}) \qquad \text{if } i > j.$$

Thus each root in Φ is a linear combination of fundamental roots with coefficients in $\mathbb{Z}$ which are either all non-negative or all non-positive. Thus we may write $\Phi = \Phi^+ \cup \Phi^-$ where Φ^+ consists of positive combinations of Π and Φ^- negative combinations.

We shall keep this example $sl_n(\mathbb{C})$ in mind to illustrate the general theory of simple Lie algebras.