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AN INTRODUCTION TO

the theory of the Riemann zeta-function

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Preface

The theory of the Riemann zeta-function and its generalisations represent one of the most beautiful developments in mathematics. The Riemann zeta-function is a meromorphic function whose properties can on the one hand be investigated by the techniques of complex analysis, and on the other yield difficult theorems concerning the integers. It is this connection between the continuous and the discrete that is so wonderful. It is the purpose of this book to explain this connection through the example of the Riemann zeta-function. The more general zeta- and L-functions will not be introduced but the reader who has studied the techniques described here should have no trouble in seeing how they apply in a more general context. This book *is* intended as an introduction; there are developments in so many directions that could have been followed, but which, in the interest of conciseness, have not.

The Riemann zeta-function belongs to 'classical' mathematics, and the development of the theory here is essentially classical. Naturally this is not the first book on this subject, nor will it be the last. The aims of this book are rather different to other books, such as the classics of Landau (Landau (1)) and Titchmarsh (Titchmarsh (2)) to which the reader interested in the finer theory will turn sooner or later, or the historical treatise of Edwards (Edwards (1)), or Ivić's book (Ivić (1)) on the most delicate modern results. It is rather to bring out the central role of the Poisson Summation Formula, and, especially, of the 'explicit formulae of prime number theory'. The knowledgeable reader will recognise the influence here of Weil. Consequently much has been omitted that might have been included simply because it did not fit into this scheme. From the point of view taken here the 'Riemann Hypothesis' takes on a central role in determining the direction of the investigations, and for this reason is discussed at much greater length than is usual. Indeed this book arose out of conversations, and then a lecture course, about the Riemann Hypothesis and Weil's point of view concerning it.

This book consists of a main part and a series of appendices which serve to summarise the background mathematics needed. In general the reader with a good undergraduate background in analysis and elementary number theory should be able to read the main part of the book without

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having to refer to the appendices too often. The appendices contain that information which does not necessarily fit in undergraduate courses. There is one point in the discussion of the Riemann Hypothesis in Chapter 5 where rather more background is required, in this case from algebraic geometry. The sections in question are marked with asterisks, and can be omitted by a reader without this background. There are a large number of exercises, of very different levels of difficulty, so that the reader can participate more actively in the mathematics.

Finally I would like to thank J. Brüdern, M. Kneser and H. Matsumoto for having read the first manuscript and for their suggestions for improvements.

Göttingen, 1986 *S.J. Patterson*

Note added at the second printing

The opportunity has been taken to correct a number of misprints. The author is particularly indebted to Professor O. McGuinness of Fordham University for communicating many of these to him.

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Notations

The notations which are used in this book are those which have become standard throughout mathematics in the last thirty years. In particular the usual language of set-theory is used – rather more consistently than is usual in analytic number-theory. Likewise functions are referred to by their name only and not through a representative variable; thus one writes ζ rather than $\zeta(s)$ when no particular argument is being discussed. If one wants to define a function by assigning particular values to an argument this will be done through the symbol $\mapsto$; thus if we define the function f to be defined by (the assignation) $x \mapsto x^2 e^x$ then $f(3) = 9e^3$ etc.

As usual $\mathbf{Z}$ will denote the ring of rational integers, $\mathbf{Q}$ the field of rational numbers, $\mathbf{R}$ the field of real numbers and $\mathbf{C}$ the field of complex numbers. Also we shall denote by $\mathbf{N}$ the set of positive (natural) integers, not including 0, and by $\mathbf{N}_0$ the set of non-negative integers, that is, including 0. Thus $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. We shall write $F^\times$ for $F - \{0\}$, where F is one of $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$ and $-$ denotes the set-theoretic difference; $F^\times$ will be called the multiplicative group of F. Also we shall write $\mathbf{R}_+$ for $\{x \in \mathbf{R} : x \geqslant 0\}$ and $\mathbf{R}_+^\times$ for $\{x \in \mathbf{R} : x > 0\} = \mathbf{R}_+^\times \subset \mathbf{R}^\times$. If $z \in \mathbf{C}$ we shall write $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ for the real and imaginary parts of z respectively.

Although it is not very important for the purposes of this book all integrations will be in the sense of Lebesgue; the Lebesgue spaces will be denoted as usual by $L^p(\cdot)$. We shall denote the closed (resp. open) interval with end-points a, b, $a < b$ by $[a, b]$ (resp. $]a, b[$) with the obvious extension to half-open intervals. We shall use the Landau O- and o-notation.

Cross-references

Inside this book cross-references are given either to a section such as §3.2 (i.e. the second section of Chapter 3) or A3.2 (i.e. the second section of Appendix 3), or to a result such as Theorem 2.5. The latter is the only theorem in that section. To enable the reader to find such cross-references easily the running section number is given at the top of each page.

References

References to the literature are given by the author's name or authors' names along with a reference number in brackets. These can then be found in the bibliography at the end of the book.

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