

Cambridge University Press  
978-0-521-49884-5 - Uniform Central Limit Theorems: Second Edition  
R. M. Dudley  
Frontmatter  
[More information](#)

---

## Uniform Central Limit Theorems

### Second Edition

This work about probability limit theorems for empirical processes on general spaces, by one of the founders of the field, has been considerably expanded and revised from the original edition. When samples become large, laws of large numbers and central limit theorems are guaranteed to hold uniformly over wide domains. The author gives a thorough treatment of the subject, including an extended treatment of Vapnik–Červonenkis combinatorics, the Ossiander L2 bracketing central limit theorem, the Giné–Zinn bootstrap central limit theorem in probability, the Bronstein theorem on approximation of convex sets, and the Shor theorem on rates of convergence over lower layers. This new edition contains proofs of several main theorems not proved in the first edition, including the Bretagnolle–Massart theorem giving constants in the Komlós–Major–Tusnády rate of convergence for the classical empirical process, Massart’s form of the Dvoretzky–Kiefer–Wolfowitz inequality with precise constant, Talagrand’s generic chaining approach to boundedness of Gaussian processes, a characterization of uniform Glivenko–Cantelli classes of functions, Giné and Zinn’s characterization of uniform Donsker classes of functions (i.e., classes for which the central limit theorem holds uniformly over all probability measures  $P$ ), and the Bousquet–Koltchinskii–Panchenko theorem that the convex hull of a uniform Donsker class is uniform Donsker.

The book will be an essential reference for mathematicians working in infinite-dimensional central limit theorems, mathematical statisticians, and computer scientists working in computer learning theory. Problems are included at the end of each chapter so the book can also be used as an advanced text.

R. M. DUDLEY is a Professor of Mathematics at the Massachusetts Institute of Technology in Cambridge, Massachusetts.

Cambridge University Press

978-0-521-49884-5 - Uniform Central Limit Theorems: Second Edition

R. M. Dudley

Frontmatter

[More information](#)

## CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

*Editorial Board:*

B. Bollobás, W. Fulton, A. Katok, F. Kirwan, P. Sarnak, B. Simon, B. Totaro

All the titles listed below can be obtained from good booksellers or from Cambridge University Press.

For a complete series listing visit [www.cambridge.org/mathematics](http://www.cambridge.org/mathematics).*Already Published*

- 99 D. Goldfeld *Automorphic forms and L-functions for the group  $GL(n, R)$*
- 100 M. B. Marcus & J. Rosen *Markov processes, Gaussian processes, and local times*
- 101 P. Gille & T. Szamuely *Central simple algebras and Galois cohomology*
- 102 J. Bertoin *Random fragmentation and coagulation processes*
- 103 E. Frenkel *Langlands correspondence for loop groups*
- 104 A. Ambrosetti & A. Malchiodi *Nonlinear analysis and semilinear elliptic problems*
- 105 T. Tao & V. H. Vu *Additive combinatorics*
- 106 E. B. Davies *Linear operators and their spectra*
- 107 K. Kodaira *Complex analysis*
- 108 T. Ceccherini-Silberstein, F. Scarabotti, & F. Tolli *Harmonic analysis on finite groups*
- 109 H. Geiges *An introduction to contact topology*
- 110 J. Faraut *Analysis on Lie groups: An introduction*
- 111 E. Park *Complex topological K-theory*
- 112 D. W. Stroock *Partial differential equations for probabilists*
- 113 A. Kirillov, Jr. *An introduction to Lie groups and Lie algebras*
- 114 F. Gesztesy et al. *Soliton equations and their algebro-geometric solutions, II*
- 115 E. de Faria & W. de Melo *Mathematical tools for one-dimensional dynamics*
- 116 D. Applebaum *Levy processes and stochastic calculus (2nd Edition)*
- 117 T. Szamuely *Galois groups and fundamental groups*
- 118 G. W. Anderson, A. Guionnet, & O. Zeitouni *An introduction to random matrices*
- 119 C. Perez-Garcia & W. H. Schikhof *Locally convex spaces over non-Archimedean valued fields*
- 120 P. K. Friz & N. B. Victoir *Multidimensional stochastic processes as rough paths*
- 121 T. Ceccherini-Silberstein, F. Scarabotti, & F. Tolli *Representation theory of the symmetric groups*
- 122 S. Kalikow & R. McCutcheon *An outline of ergodic theory*
- 123 G. F. Lawler & V. Limic *Random walk: A modern introduction*
- 124 K. Lux & H. Pahlings *Representations of groups*
- 125 K. S. Kedlaya *p-adic differential equations*
- 126 R. Beals & R. Wong *Special functions*
- 127 E. de Faria & W. de Melo *Mathematical aspects of quantum field theory*
- 128 A. Terras *Zeta functions of graphs*
- 129 D. Goldfeld & J. Hundley *Automorphic representations and L-functions for the general linear group, I*
- 130 D. Goldfeld & J. Hundley *Automorphic representations and L-functions for the general linear group, II*
- 131 D. A. Craven *The theory of fusion systems*
- 132 J. Vaananen *Models and games*
- 133 G. Malle & D. Testerman *Linear algebraic groups and finite groups of Lie type*
- 134 P. Li *Geometric analysis*
- 135 F. Maggi *Sets of finite perimeter and geometric variational problems*
- 136 M. Brodmann & R. Y. Sharp *Local cohomology (2nd Edition)*
- 137 C. Muscalu & W. Schlag *Classical and multilinear harmonic analysis, I*
- 138 C. Muscalu & W. Schlag *Classical and multilinear harmonic analysis, II*
- 139 B. Helffer *Spectral theory and its applications*
- 140 M. Pemantle & M. Wilson *Analytic combinatorics in several variables*

Cambridge University Press

978-0-521-49884-5 - Uniform Central Limit Theorems: Second Edition

R. M. Dudley

Frontmatter

[More information](#)

---

# Uniform Central Limit Theorems

Second Edition

R. M. DUDLEY

*Massachusetts Institute of Technology*



**CAMBRIDGE**  
UNIVERSITY PRESS

Cambridge University Press

978-0-521-49884-5 - Uniform Central Limit Theorems: Second Edition

R. M. Dudley

Frontmatter

[More information](#)CAMBRIDGE  
UNIVERSITY PRESS

32 Avenue of the Americas, New York, NY 10013-2473, USA

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

[www.cambridge.org](http://www.cambridge.org)Information on this title: [www.cambridge.org/9780521738415](http://www.cambridge.org/9780521738415)

© R. M. Dudley 1999, 2014

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First edition published 1999

First paperback edition 2008

Second edition 2014

Printed in the United States of America

*A catalog record for this publication is available from the British Library.**Library of Congress Cataloging in Publication data*

Dudley, R. M. (Richard M.)

Uniform central limit theorems / R. M. Dudley, Massachusetts Institute of Technology. –  
Second edition.

pages cm – (Cambridge studies in advanced mathematics)

Includes bibliographical references and index.

ISBN 978-0-521-49884-5 (hardback) – ISBN 978-0-521-73841-5 (paperback)

1. Central limit theorem. I. Title.

QA273.67.D84 2014

519.2–dc23 2013011303

ISBN 978-0-521-49884-5 Hardback

ISBN 978-0-521-73841-5 Paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party Internet Web sites referred to in this publication and does not guarantee that any content on such Web sites is, or will remain, accurate or appropriate.

Cambridge University Press

978-0-521-49884-5 - Uniform Central Limit Theorems: Second Edition

R. M. Dudley

Frontmatter

[More information](#)

---

To Liza

Cambridge University Press

978-0-521-49884-5 - Uniform Central Limit Theorems: Second Edition

R. M. Dudley

Frontmatter

[More information](#)

---

## Contents

<i>Preface to the Second Edition</i>	<i>page xi</i>
<b>1 Donsker's Theorem and Inequalities</b>	<b>1</b>
1.1 Empirical Processes: The Classical Case	6
1.2 Metric Entropy and Capacity	7
1.3 Inequalities	9
1.4 *Proof of the Bretagnolle–Massart Theorem	15
1.5 The DKW Inequality in Massart's Form	39
<b>2 Gaussian Processes; Sample Continuity</b>	<b>61</b>
2.1 General Empirical and Gaussian Processes	61
2.2 Some Definitions	62
2.3 Bounds for Gaussian Vectors	67
2.4 Inequalities for Gaussian Distributions	73
2.5 Sample Boundedness	82
2.6 Gaussian Measures and Convexity	85
2.7 Regularity of the Isonormal Process	88
2.8 A Metric Entropy Condition for Continuity	94
2.9 Gaussian Concentration Inequalities	100
2.10 Generic Chaining	108
2.11 Homogeneous and Quasi-Homogeneous Sets in $H$	117
2.12 Sample Continuity and Compactness	121
2.13 Two-Series and One-Series Theorems	125
<b>3 Definition of Donsker Classes</b>	<b>133</b>
3.1 Definitions: Convergence in Law	133
3.2 Measurable Cover Functions	137
3.3 Almost Uniform, Outer Probability Convergence	143

viii	<i>Contents</i>	
	3.4 Perfect Functions	145
	3.5 Almost Surely Convergent Realizations	149
	3.6 Conditions Equivalent to Convergence in Law	154
	3.7 Asymptotic Equicontinuity	159
	3.8 Unions of Donsker Classes	162
	3.9 Sequences of Sets and Functions	163
	3.10 Donsker Classes and Sequential Limits	168
	3.11 Convex Hulls of Donsker Classes	168
<b>4</b>	<b>Vapnik–Červonenkis Combinatorics</b>	<b>175</b>
	4.1 Vapnik–Červonenkis Classes of Sets	175
	4.2 Generating Vapnik–Červonenkis Classes	179
	4.3 *Maximal Classes	183
	4.4 *Classes of Index 1	185
	4.5 *Combining VC Classes	192
	4.6 Probability Laws and Independence	200
	4.7 VC Properties of Function Classes	204
	4.8 Classes of Functions and Dual Density	205
<b>5</b>	<b>Measurability</b>	<b>213</b>
	5.1 Sufficiency	215
	5.2 Admissibility	222
	5.3 Suslin Properties and Selection	229
<b>6</b>	<b>Limit Theorems for VC-Type Classes</b>	<b>239</b>
	6.1 Glivenko–Cantelli Theorems	239
	6.2 Glivenko–Cantelli Properties for Given $P$	247
	6.3 Pollard’s Central Limit Theorem	251
	6.4 Necessary Conditions for Limit Theorems	260
<b>7</b>	<b>Metric Entropy with Bracketing</b>	<b>269</b>
	7.1 The Blum–DeHardt Theorem	269
	7.2 Bracketing Central Limit Theorems	274
	7.3 The Power Set of a Countable Set	279
<b>8</b>	<b>Approximation of Functions and Sets</b>	<b>284</b>
	8.1 Introduction: The Hausdorff Metric	284
	8.2 Spaces of Differentiable Functions and Sets	287
	8.3 Lower Layers	300
	8.4 Metric Entropy of Classes of Convex Sets	305
<b>9</b>	<b>Two Samples and the Bootstrap</b>	<b>319</b>
	9.1 The Two-Sample Case	319



Cambridge University Press

978-0-521-49884-5 - Uniform Central Limit Theorems: Second Edition

R. M. Dudley

Frontmatter

[More information](#)

<i>Contents</i>	ix
9.2 A Bootstrap CLT	323
9.3 Other Aspects of the Bootstrap	345
<b>10 Uniform and Universal Limit Theorems</b>	<b>348</b>
10.1 Uniform Glivenko–Cantelli Classes	348
10.2 Universal Donsker Classes	360
10.3 Metric Entropy of Convex Hulls in Hilbert Space	366
10.4 Uniform Donsker Classes	372
10.5 Universal Glivenko–Cantelli Classes	388
<b>11 Classes Too Large to Be Donsker</b>	<b>391</b>
11.1 Universal Lower Bounds	391
11.2 An Upper Bound	393
11.3 Poissonization and Random Sets	395
11.4 Lower Bounds in Borderline Cases	400
11.5 Proof of Theorem 11.10	410
<b>Appendices</b>	
A Differentiating under an Integral Sign	417
B Multinomial Distributions	424
C Measures on Nonseparable Metric Spaces	427
D An Extension of Lusin’s Theorem	430
E Bochner and Pettis Integrals	432
F Nonexistence of Some Linear Forms	437
G Separation of Analytic Sets	440
H Young–Orlicz Spaces	443
I Versions of Isonormal Processes	446
<i>Bibliography</i>	449
<i>Notation Index</i>	463
<i>Author Index</i>	465
<i>Subject Index</i>	468

Cambridge University Press

978-0-521-49884-5 - Uniform Central Limit Theorems: Second Edition

R. M. Dudley

Frontmatter

[More information](#)

---

Cambridge University Press

978-0-521-49884-5 - Uniform Central Limit Theorems: Second Edition

R. M. Dudley

Frontmatter

[More information](#)

## Preface to the Second Edition

This book developed out of some topics courses given at M.I.T. and my lectures at the St.-Flour probability summer school in 1982. The material of the book has been expanded and extended considerably since then. At the end of each chapter are some problems and notes on that chapter.

Starred sections are not cited later in the book except perhaps in other starred sections. The first edition had several double-starred sections in which facts were stated without proofs. This edition has no such sections.

The following, not proved in the first edition, now are: (i) for Donsker's theorem on the classical empirical process  $\alpha_n := \sqrt{n}(F_n - F)$ , and the Komlós–Major–Tusnády strengthening to give a rate of convergence, the Bretagnolle–Massart proof with specified constants; (ii) Massart's form of the Dvoretzky–Kiefer–Wolfowitz inequality for  $\alpha_n$  with optimal constant; (iii) Talagrand's generic chaining approach to boundedness of Gaussian processes, which replaces the previous treatment of majorizing measures; (iv) characterization of uniform Glivenko–Cantelli classes of functions (from a paper by Dudley, Giné, and Zinn, but here with a self-contained proof); (v) Giné and Zinn's characterization of uniform Donsker classes of functions; (vi) its consequence that uniformly bounded, suitably measurable classes of functions satisfying Pollard's entropy condition are uniformly Donsker; and (vii) Bousquet, Koltchinskii, and Panchenko's theorem that a convex hull preserves the uniform Donsker property.

The first edition contained a chapter on invariance principles, based on a 1983 paper with the late Walter Philipp. Some techniques introduced in that paper, such as measurable cover functions, are still used in this book. But I have not worked on invariance principles as such since 1983. Much of the work on them treats dependent random variables, as did parts of the 1983 paper which Philipp contributed. The present book is mainly about the i.i.d. case. So I suppose the chapter is outdated, and I omit it from this edition.

Cambridge University Press

978-0-521-49884-5 - Uniform Central Limit Theorems: Second Edition

R. M. Dudley

Frontmatter

[More information](#)

For useful conversations and suggestions on topics in the book I'm glad to thank Kenneth Alexander, Niels Trolle Andersen, the late Miguel Arcones, Patrice Assouad, Erich Berger, Lucien Birgé, Igor S. Borisov, Donald Cohn, Yves Derrienic, Uwe Einmahl, Joseph Fu, Sam Gutmann, David Haussler, Jørgen Hoffmann-Jørgensen, Yen-Chin Huang, Vladimir Koltchinskii, the late Lucien Le Cam, David Mason, Pascal Massart, James Munkres, Rimas Norvaiša, the late Walter Philipp, Tom Salisbury, the late Rae Shortt, Michel Talagrand, Jon Wellner, He Sheng Wu, Joe Yukich, and Joel Zinn. I especially thank Denis Chetverikov, Peter Gaenssler and Franz Strobl, Evarist Giné, and Jinghua Qian, for providing multiple corrections and suggestions. I also thank Xavier Fernique (for the first edition), Evarist Giné (for both editions), and Xia Hua (for the second edition) for giving or sending me copies of expositions.

### Notes

Throughout this book, all references to “RAP” are to the author's book *Real Analysis and Probability*, second edition, Cambridge University Press, 2002.

Also, “ $A := B$ ” means  $A$  is defined by  $B$ , whereas “ $A =: B$ ” means  $B$  is defined by  $A$ .