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Vector Bundles in Algebraic Geometry

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Edited by

N.J. Hitchin
University of Cambridge

P.E. Newstead
University of Liverpool

W.M. Oxbury
University of Durham

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Introduction

There are few areas of pure mathematics which have profited more from the influx of new ideas from other disciplines than the subject of this book. The study of vector bundles over algebraic varieties has been stimulated by successive waves of migrant concepts over the last few years, largely coming from mathematical physics. It nevertheless also retains its roots in old questions concerning subvarieties of projective space, and this is a continuing source of challenging problems. The 1993 Durham Symposium *Vector Bundles in Algebraic Geometry*, sponsored by the LMS and SERC, had as its aim the goal of bringing together the leading researchers in the field to explore further these interactions: to see how old problems would yield to new techniques, and to present new opportunities for the already highly developed subject of algebraic geometry.

The present book is not, however, simply the Proceedings of that Symposium. Its purpose is certainly to reflect what was said and done there, but we hope that it also presents to the mathematical world an overview of the key areas of research involving vector bundles. Some of the principal speakers have been encouraged to expand their talks to give surveys of their respective areas. The reader can thus find here not only reports of recent progress, but also a perspective on where the new ideas have come from, what they are doing at the moment, and what they might be capable of in the future.

The incursions from mathematical physics and differential geometry have taken several forms, and all are represented here. Probably the first one began twelve years ago with Donaldson's proof of the theorem of Narasimhan and Seshadri using gauge-theoretic methods and ideas from symplectic geometry. The legacy of that approach can be seen in the article of Bradlow, Daskalopoulos, Garcia-Prada and Wentworth, where the concept of moment map gives on the one hand a secure analytical foundation to the construction of moduli spaces of bundles with extra structure, and also its link with the algebraic geometric viewpoint of Geometric Invariant Theory. One of the remarkable features of the symposium was the way in which various forms of 'augmented structure' were appearing spontaneously in the content of many talks, as a response to different types of problems. This particular paper gives a statement of the state of the art in this general area. One of these structures, arising in a purely algebro-geometric setting, is what Le Potier calls a 'coherent system'. The discussion of these systems, including some applications, forms part of his contribution to this book. In particular, he uses coherent systems together with the concept of determinant bundle (which itself has already arisen in several contexts) to determine the Picard groups of some moduli spaces.

The input of gauge theory into the subject is not confined to symplectic geometry, and Donaldson's polynomial invariants, yielding information on the differentiable structure of algebraic surfaces, form an extremely active area of research. It relies of course on the study of moduli spaces of stable holomorphic bundles, but by embedding the subject in the more general one of instantons on four-manifolds, it lends it an extra degree of flexibility. The 'expected dimensions' acquire a reality of interpretation which can be put to use to prove very deep results. There are two contributions here in this field. The first is by Donaldson himself on Floer homology and the second by Tyurin on the impact of Donaldson polynomials in algebraic geometry. Donaldson's paper includes some concrete calculations for elliptic surfaces and a preview of the 'quantum cohomology' of the moduli space of bundles on a curve, surely a topic for rapid development in the next few years, while Tyurin advances the 'Jacobian' of an algebraic surface (the Gieseker closure of the component of the moduli space of bundles which contains the cotangent bundle) as an essential tool for the study of surface geometry. It is clear by now for various reasons that moduli spaces of vector bundles have an exceedingly rich internal geometry which rivals the classical theory of the Jacobian of a curve. Tyurin's terminology is quite apposite. In a similar vein, the paper of Balaji and Vishwanath discusses the analogues of the Picard bundles over moduli spaces for curves.

In recent years a second influx of ideas from physics has come from conformal field theory and in particular the challenge of understanding the Verlinde formulae which describe the dimension of the space of sections of an ample line bundle over the moduli space of bundles on a curve. There are many aspects to this problem, and by now many proofs using different methods. Here, we restrict ourselves to two discussions of the topic. In one, Ueno describes for algebraic geometers the construction of 'conformal blocks' from the physicists' description of conformal field theory. In the other contribution, Szenes discusses the structure of the formulae themselves, and in particular in the case of groups other than SL_2. The whole subject is very broad, and a complete picture requires the establishment of rigorous links between different languages. It happened that, in the course of the symposium, key results providing some of these were announced by Narasimhan, Beauville and Laszlo but these will appear elsewhere.

In any mathematical discipline, it is unwise to concentrate unduly on the general at the expense of the particular. Algebraic geometry in particular is a mature subject which is blessed with a wealth of beautiful examples, and techniques which make even non-generic objects subject to close analysis. Within the realm of holomorphic vector bundles, the Horrocks-Mumford bundle is highly distinguished; the only known indecomposable rank 2 bundle on $\mathbb{P}^4$, with 15,000 symmetries and closely related to modular surfaces, it stands out like the icosahedron as an object of endless fascination. Hulek gives here a survey of what is known about this bundle and its relationship with other areas of mathematics since its discovery in 1972. A section of the Horrocks-Mumford bundle gives an abelian surface in $\mathbb{P}^4$ which is not a complete intersection. This general problem in $\mathbb{P}^4$ and $\mathbb{P}^5$ is considered by Decker and Popescu in their attack on the classification of codimenson 2 subvarieties. Although a subject of some considerable antiquity, there are now new results emanating from computer algebra working on Beilinson's spectral sequence. The basic information about the moduli space of stable bundles, its dimenson and smoothness, is given by the sheaf cohomology groups $H^i(X, Ad(E))$ for $0 \leq i \leq 2$. A bundle for which the groups vanish for all i is called *exceptional* and is the subject of Drézet's paper. Exceptional bundles are building blocks for semi-stable sheaves on projective spaces, and the author uses them to give information about the moduli spaces for projective spaces of arbitrary dimension.

The Durham symposium was a successful forum for discussing all these new approaches to the subject of vector bundles in algebraic geometry. We hope that this book will make available to a wider mathematical public the essential ideas in this rapidly developing area.

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