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On the Deformation Theory of Moduli Spaces of Vector Bundles

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§0 Introduction

This article is of the nature of a discussion of various results in the deformation theory of moduli spaces of vector bundles and the techniques involved in their proofs. We have concentrated mostly on the conceptual points of proofs and largely ignored all technical and computational details with needed references to enable the interested reader to fill in the gaps.

The layout of the article is as follows: in section 1, we discuss results on the deformations of the moduli spaces of rank 2 vector bundles. The topic of section 2 concerns a study of intermediate Jacobians of these objects with a view to getting Torelli type theorems for moduli spaces and finally, in section 3 we discuss the deformations of Picard bundle on the moduli space of vector bundles.

§1 Deformations of the moduli space

Let C be a smooth projective curve of genus $g \geq 3$ defined over the complex number field $\mathbb{C}$. Let ξ be a line bundle on C. Let

$$M_\xi := M_C(2, \xi)$$

denote the moduli space of semi-stable vector bundles of rank 2 and determinant isomorphic to ξ on C. In this section we consider M_ξ for $\xi \cong \mathcal{O}_C$ and $\xi \cong \mathcal{O}_C(x_0)$, $x_0 \in C$. These we abbreviate by M_0 and M_1 respectively. Both M_0 and M_1 are projective varieties which are unirational and normal of dimension $3g-3$. Further, M_1 is a smooth variety. The theorem of Narasimhan and Ramanan computes the cohomology of the tangent bundle of M_1. More precisely,

Theorem 1.1. (cf [NR])[1] *If T_{M_1} denotes the tangent bundle of M_1, we have*

$$h^i(M_1, T_{M_1}) := \dim H^i(M_1, T_{M_1}) = \begin{cases} 3g-3 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Remark 1.2.

(a) Note that the group $H^0(M_1, T_{M_1})$, of infinitesimal automorphisms of M_1, is trivial implies that the group of automorphisms, $\mathrm{Aut}(M_1)$, of M_1 is discrete. But as M_1 is known to be canonically polarised (i.e. the canonical class is negatively ample), $\mathrm{Aut}(M_1)$ is an algebraic group and hence is finite.

(b) The number of moduli, $h^1(M_1, T_{M_1})$, of M_1 (cf. [KS]) is equal to the number of moduli of C. This intuitively means that any small variation of M_1 is again a moduli space of vector bundles on a variation of C.

(c) Again, because the canonical class of M_1 is negatively ample, Nakano's vanishing theorem (cf. [KS]) gives the vanishing of $h^i(M_1, T_{M_1})$ for $i \geq 2$. So it suffices, to prove the Theorem 1.1, to compute h^0 and h^1.

The proof of Theorem 1.1 uses correspondence techniques.[2] That is, let V be the universal bundle on $M_1 \times C$. Let ad V denote the bundle of traceless endomorphisms of V. Let p and q denote the projections from $M_1 \times C$ onto M_1 and C respectively. We then have two Leray spectral sequences

$$\begin{array}{ccc} H^{i+j}(M_1 \times C, \mathrm{ad}\ V) & \Leftarrow & H^i(M_1, R^j p_*(\mathrm{ad}\ V)) \\ \Uparrow & & \\ H^i(C, R^j q_*(\mathrm{ad}\ V)) & & \end{array}$$

[1]Of course, the theorem is true more generally in all rank n and degree d such that $(n,d) = 1$ and curves of all genera $g > 1$.

[2]We should add here that there is a different proof of Theorem 1.1, due to N.Hitchin, available now (cf. [H]).

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On the one hand, deformation theory of stable bundles on a curve yields

$$R^j p_*(\text{ad } V) \cong \begin{cases} 0 & \text{if } i \neq 1 \\ T_{M_1} & \text{if } i = 1. \end{cases}$$

When this is fed into the Leray spectral sequence arising from the projection p, we get

$$H^i(M_1, T_{M_1}) \cong H^i(M_1 \times C, \text{ad } V). \tag{1.1}$$

On the other hand, if we could somehow study the deformation theory of the family $\{V_x\}_{x \in C}$ (where $V_x := V|_{M_1 \times \{x\}}$) of bundles on M_1 parametrised by C, we could hope to connect $H^*(M_1 \times C, \text{ad } V)$ with $H^*(C, T_C)$ and try to compute $h^i(M_1, T_{M_1})$. This is provided by

Proposition 1.3. (cf. [NR], [S1]) *Let* $\text{ad}_x V := \text{ad}(V_x)$, $x \in C$. *Then we have:*

(a) $\dim H^i(M_1, \text{ad}_x V) = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i = 0, 2 \end{cases}$

(b) The Kodaira-Spencer map (cf. [KS], [NR]) *for the family* $\{V_x\}_{x \in C}$

$$\rho_x : T_{C,x} \longrightarrow H^1(M_1, \text{ad}_x V)$$

is an isomorphism for all $x \in X$.

Proof of Theorem 1.1: Note that Proposition 1.3 has as a consequence:

$$R^j q_*(\text{ad } V) \simeq \begin{cases} 0 & \text{if } i \neq 1 \\ T_C & \text{if } i = 1. \end{cases}$$

This, together with the Leray spectral sequence arising from the projection q, gives

$$H^i(C, T_C) \cong H^i(M_1 \times C, \text{ad } V). \tag{1.2}$$

Now put (1.1) and (1.2) together to complete the proof of Theorem 1.1.

The proof of Proposition 1.3 uses a construction which has come to be familiar as "Hecke Correspondence" (cf. [NR], [MS], [B]). Briefly, if $P := \mathbf{P}(V_x)$, it can be shown, using "elementary transformations" that P parametrises a family $\{W_t\}_{t \in P}$ of rank 2, trivial determinant semistable vector bundles on C. Consider

$$\begin{array}{ccc} & P & \\ {}^{\psi}\swarrow & & \searrow^{\phi} \\ M_1 & & M_0 \end{array}$$

where ψ is the canonical projection and ϕ is the characteristic map given by the family $\{W_t\}_{t\in P}$. It is clear that ϕ is a P^1-bundle when restricted to the open set, U, of stable bundles in M_0. We then have

(a) The codimension of $P - \phi^{-1}(U)$ in P is at least $g-1$ where g is the genus of the curve.

(b) Let T_ψ and T_ϕ denote the relative tangent sheaves of the maps ψ and ϕ respectively. We have, on $\phi^{-1}(U)$, $T_\psi \simeq T^*_\phi$.

These facts, together with the Leray spectral sequence and a repeated application of Hartog type theorem for cohomology (see [S1] for details), give

$$H^i(M_1, \mathrm{ad}_x V) \cong H^{i-1}(M_0, \mathcal{O}) \quad 0 \le i \le 2.$$

But M_0 is known to be unirational and complete. This proves (a). For a proof of (b) see [NR] or [S1].

Let N be the desingularisation of M_0 constructed in [S2]. Let PV_4 denote the category of parabolic semi-stable bundles (V, Δ), where V is of rank 4 and $\det V \simeq \mathcal{O}_C$, and Δ is a parabolic structure at the marked point $x_0 \in C$ with small weights (α_1, α_2). Then N parametrises isomorphism classes of vector bundles in P_4 such that End V, the endomorphism algebra, is a specialisation of the (2×2)-matrix algebra $\mathcal{M}_2$.

Theorem 1.4. (cf. [BV1]) *If T_N denotes the tangent bundle of N, we have*

$$h^i(N, T_N) = \begin{cases} 3g-3 & \text{if } i = 1 \\ 0 & \text{if } i = 0, 2 \end{cases}$$

Before proceeding with an idea of proof of Theorem 1.4, we collect some facts about the variety N we need for the proof (cf. [BS], [BV1]).

(a) The variety N represents a natural moduli functor and hence there exists a universal family of rank 4 bundles on $C \times N$.

(b) There also exists a family of quadratic forms $\{Q_t\}_{t\in N}$ on a fixed 3-dimensional vector space parametrised by N. This stratifies N into subvarieties $\{N_i\}_{i=1}^3$ defined by

$$N_i = \{t \in N \mid \mathrm{rank}\, Q_t \le 3 - i\}.$$

Thus $N \supset N_1 \supset N_2 \supset N_3$. If we denote $N - N_2$ by Z and $N_1 - N_2$ by Y, then Y is a smooth divisor in Z with $Z - Y \simeq U$ (U is the open set of stable bundles in M_0). Further, Y is a $\mathbf{P}^{g-2} \times \mathbf{P}^{g-2}$ - bundle on $K - K_0$; where K is the Kummer variety of $\dim g$ associated to the Jacobian of C, K_0 is its singular locus. The normal bundle, n_Y, of Y in Z, is isomorphic to $\mathcal{O}(-1,-1)$ when restricted to the fibres of Y over $K - K_0$.

(c) Finally, the codimension of N_2 in N is 3.

We can now outline very briefly the key steps involved in proving Theorem 1.4.

Step 1. By a switching trick in the Hecke correspondence, we can compute the dimension of $H^i(U, T_U)$. In fact

$$h^i(U,T_U) = \begin{cases} 0 & i = 0,2 \\ 3g-3 & i = 1. \end{cases}$$

Step 2. By the use of a Hartog type argument, using part (c) above, we see that it is enough to compute $h^i(Z, T_Z)$.

Step 3. $H^i(Z, T_Z) \cong H^i(U, T_U) \quad i = 0, 1, 2.$

Here we need the following cohomological result (cf. [G]).

$$\varinjlim_n H^i(Z, \mathcal{O}_Z(nY) \otimes T_Z) \cong H^i(U, T_U)$$

where $Y \subset Z$ is the divisor and $U = Z - Y$.

Consider now the following exact sequence:

$$A_k : 0 \longrightarrow \mathcal{O}_Z((k-1)Y) \longrightarrow \mathcal{O}_Z(kY) \longrightarrow n_Y^k \longrightarrow 0$$

for all $k > 0$. Here n_Y is the normal bundle of Y in Z.

Using the explicit description of n_Y in (b) above, we prove that

$$H^i(Z, T_Z \otimes n_Y^k) = 0, \quad i = 0, 1, 2 \quad \forall k > 0.$$

Then, by step 3 and the long exact sequence of A_k, we get, for all $k > 0$

$$H^i(Z, T_Z \otimes \mathcal{O}(kY)) \simeq H^i(Z, T_Z \otimes \mathcal{O}((k-1)Y))$$
$$\Longrightarrow H^i(Z, T_Z) \simeq H^i(U, T_U).$$

§2 Intermediate Jacobians, Torelli-type theorems

This section, as the title indicates, deals with Torelli type theorems for the moduli spaces $M := M_1$ and N, the natural desingularisation of M_0. We sometimes write M_C and N_C to stress the dependence of these spaces on the curve C.

The Weil-Griffiths intermediate Jacobian associated to the third cohomology group of a unirational variety V is an interesting invariant (cf. [Gr]) and is especially suited to study Torelli type theorems for the moduli spaces of vector bundles. The key concept involved here is the so called 'Weil map'. We recall in brief the definition of the Weil map.

Let V be a smooth projective unirational variety and let T be a parameter space which we assume to be a smooth projective variety. Let A be an algebraic cycle on $V \times T$ of codimension 2. Then we have its fundamental class $\alpha \in H^4(V \times T, \mathbf{Z})$. Assume $H^2(V, \mathbf{Z})_{tor} = 0$. Then the (1,3)-Kunneth component

$$\alpha_{1,3} \in H^1(V, \mathbf{Z}) \otimes H^3(T, \mathbf{Z})$$

gives a homomorphism

$$\alpha_{1,3} : H_1(T, \mathbf{Z}) \longrightarrow H^3(V, \mathbf{Z});$$

from which we get a map of real tori

$$\phi_A : \frac{H_1(T, \mathbf{R})}{H_1(T, \mathbf{Z})} \longrightarrow \frac{H^3(V, \mathbf{R})}{H^3(V, \mathbf{Z})}. \tag{2.3}$$

The real vector spaces $H^3(V, \mathbf{R})$ and $H_1(T, \mathbf{R})$ are given complex structures through the C-operator in Hodge theory. Now the fact that the form α is of Hodge type $(2,2)$, since it comes from an algebraic cycle, implies that ϕ_A is actually a holomorphic map between the complex tori $Alb(T) := \frac{H_1(T,\mathbf{R})}{H_1(T,\mathbf{Z})}$, the Albanese of T and $J^2(V) := \frac{H^3(V,\mathbf{R})}{H^3(V,\mathbf{Z})}$, the 2nd intermediate Jacobian of V. This is termed the Weil map and we again denote the map by ϕ_A. One of the very important properties of this map is its functorial behaviour with respect to maps between parameter spaces (cf. [L]). We will have occasion to return to this point later in section 3. To turn these compex tori into Abelian varieties, let L be an ample line bundle on V and if w is the Kälher form on V associated with L, define a pairing

$$H^3(V,\mathbb{C}) \times H^3(V,\mathbb{C}) \longrightarrow \mathbb{C}$$

$$(\alpha,\beta) \longrightarrow \int_V w^{n-3} \wedge \alpha \wedge \beta \qquad (n = \dim .V).$$

It can be shown that this pairing satisfies Hodge-Riemann conditions turning $J^2(V)$ into a polarised Abelian variety. Note that we have tacitly assumed that all classes in $H^3(V,\mathbb{R})$ are primitive. This is true because of our assumption that V is unirational. This assumption is satisfied for the examples we shall consider. We also let Ψ_L to denote the ample line bundle on $J^2(V)$ defined by the above pairing and often refer to Ψ_L as the polarisation on $J^2(V)$ induced from L.

In the context of the moduli spaces of vector bundles, if W is a universal bundle on $C \times M$, its second chern class $c_2(W) \in H^2(C \times M, \mathbb{Z})$ gives the Weil map

$$\phi_W : Alb(C) \to J^2(M).$$

We remark that the unirationality of M ensures that this map is independent of the choice of the universal bundle on $C \times M$. Similar considerations apply to the variety N: if E denotes *the* rank 4 bundle on $C \times N$, we get the Weil map

$$\phi_E : Alb(C) \to J^2(N).$$

We then have

Theorem 2.1. (cf. [MN],[NR],[B]) *The Weil maps*

(a) $\phi_W : Alb(C) \to J^2(M)$ *is an isomorphism of Abelian varieties .*

(b) $\phi_E : Alb(C) \to J^2(N)$ *is an isogeny of degree* 2^g.

The proof of the theorem is rather technical and we concentrate only on its consequences. One natural question in the context of the Torelli type theorem for moduli spaces is whether the polarisation on $J^2(M)$ (resp. $J^2(N)$) induced from M (resp. N) is independent of the choice of an ample line bundle on M (resp. N). Before we discuss this we make a definition.

Definition 2.2. *Let A be an Abelian variety and L_1 and L_2 be two line bundles on A. We say that L_1 is equivalent to L_2 (written $L_1 \equiv L_2$) if a* *power of* L_1 *is algebraically equivalent to a power of* L_2. *Further, if* A *(resp.* A'*) is given the polarisation* L *(resp.* L'*), we say that a map* $f : A \to A'$ *of Abelian varieties is polarisation preserving if* $f^* L' \equiv L$ *on* A.

In the case of M, since its anticanonical class is known to be ample, there is a canonical choice of a polarisation on $J^2(M)$; namely, the one induced from the dual of the canonical bundle. But, unfortunately, this fact is not known for the variety N ($Pic(N) \simeq \mathbf{Z} \oplus \mathbf{Z}$). So, let C_0 be a smooth curve for which the Neron-Severi group of its Jacobian, $NS(J(C_0))$, is isomorphic to $\mathbf{Z}$. It is well known that such curves exist. If $\Psi_1(C_0)$ and $\Psi_2(C_0)$ are two polarisations on $J^2(N)$ induced from two ample line bundles on N_{C_0}, we have, from part (b) of Theorem 2.1,

$$\Psi_1(C_0) \equiv \Psi_2(C_0).$$

Now, if C is any smooth curve and L_1 and L_2 are ample line bundles on N_C, connect C to C_0 in a holomorphic one parameter family $\{C_t\}$. It can then be shown, from the nature of construction of N, that L_1 and L_2 can be spread out to the whole of the family. Then, $\phi_E^* \Psi_2(C_t)$ and $\phi_E^* \Psi_1(C_t)$ are sections, over the family, of the local system formed from the cohomology groups $\{H^2(N_{C_t}, \mathbf{Z})\}$. And at curve C_0, some powers of these sections agree—thereby implying that the same powers of $\phi_E^* \Psi_1(C)$ and $\phi_E^* \Psi_2(C)$ agree on $J^2(N_C)$. That is, the equivalence class of the induced polarisation on $J^2(N)$ is independent of the choice of an ample line bundle on N. In what follows we always identify $Alb(C)$ with the Jacobian, $J(C)$, of C and give the natural polarisation $\Theta(C)$, afforded by the "theta divisor", on $Alb(C)$. Note that the above argument also proves that the map ϕ_E is polarisation preserving. A similar statement holds for the map ϕ_W.

Once these preliminary details regarding polarisations are fixed, a Torelli type theorem for the space N (along similar lines for the space M) can be proved as follows: suppose that the moduli spaces N_{C_1} and N_{C_2} are isomorphic via an isomorphism f and let E_1 (resp. E_2) be the universal rank 4 bundle on $C \times N_1$ (resp. $C \times N_2$). If $\overline{f}$ is the induced isomorphism between $J^2(N_1)$ and $J^2(N_2)$, using the explicit nature of the isogenies ϕ_{E_i}, one can show the existence of a commutative diagram

$$\begin{array}{ccc} J^2(N_{C_1}) & \xrightarrow{\overline{f}} & J^2(N_{C_2}) \\ \phi_{E_1}\uparrow & & \uparrow\phi_{E_2} \\ Alb(C_1) & \xrightarrow{u} & Alb(C_2) \end{array}$$

where u is again an isomorphism. And because ϕ_{E_i} and $\overline{f}$ are polarisation preserving, it easily follows, from the commutativity of the diagram, that $u^*\Theta(C_2) \equiv \Theta(C_1)$. But then, by the classical Torelli theorem, we find that the curves C_1 and C_2 are isomorphic.

§3 Deformations of the Picard bundle

In this section we let M denote the moduli space of stable bundles on the curve C of rank 2 and determinant isomorphic to a fixed line bundle ξ of degree d. We consider only the case where d is a fixed odd integer greater than $4g-3$. We also let W denote a universal family on $M \times C$. Let p and q stand for the canonical projections from $M \times C$ to M and C respectively. Then the direct image sheaf $\mathcal{U} := p_*(W)$ is a locally free sheaf and is referred to as a Picard bundle on M. Note that this depends on the choice of the universal bundle on $M \times C$. So we work with a fixed choice of a universal family. This construction also comes with an obvious family of deformations of the Picard bundle, $\{\mathcal{W}(j)\}_{j\in J}$, parameterised by the Jacobian , $J := J(C)$, of C. Namely, for $j \in J(C)$, set

$$\mathcal{W}(j) := p_*(W \times q^* L_j)$$

(here L_j is the line bundle corresponding to $j \in J$. The main theorem of this section is

Theorem 3.1. (cf. [BV2]) *For a smooth curve C, without automorphisms, of genus g, $g > 2$, we have*

(a) $\dim H^i(M, \mathrm{ad}\mathcal{W}) = \begin{cases} g & \text{if } i = 1 \\ 0 & \text{if } i = 0, 2 \end{cases}$

(b) The g-dimensional family $\{\mathcal{W}(j)\}_{j\in J}$, defined above, is injective.

Remark 3.2. A detailed study of Picard bundles, $\mathcal{P}_d$, on $J^{(d)}$, the component of $Pic(C)$ parametrising degee d ($d > 2g-1$) line bundles, has been studied extensively. See [M], [K1], [K2] for questions regarding the topology, deformations and cohomology of these objects.

Before proceeding with the idea of proof, we spend some time on the results of Thaddeus (cf. [T]) which realises the projective bundle of the

Picard bundle, $\mathbf{P}(\mathcal{W})$, as an end product of a series of blow-ups and blow-downs starting with a projective space. More precisely, let P_i ($i \in \mathbf{Z}; 0 \leq i \leq \frac{d-1}{2}$) denotes the moduli space of pairs (V, s); where V is a point in M and $s \in \mathbf{P}(H^0(C,V))$. The pair (V,s) is assumed to be stable with respect to a weight $\alpha \in \{max(0, \frac{d}{2} - i - 1), \frac{d}{2} - i\}$. These moduli spaces exist as smooth projective varieties and carry universal families. We summarise the results that we need in this work as follows.

(a) For $i = 0$, the space P_0 can be identified with the projective space $\mathbf{P}(H^0(K_C \otimes \xi)^*)$; where K_C is the canonical bundle of the curve C. Also, the space P_1 can be obtained from P_0 by blowing up the space P_0 along the curve C embedded in P_0 via the complete linear series $|K_C \otimes \xi|$. This blow up map we denote by ψ_1.

(b) For $i > 1$, one can pass from P_{i-1} to P_i by a blow up and a blow down. More precisely, we have a diagram

$$\begin{array}{ccc} \tilde{P}_i & \xrightarrow{\phi_i} & P_i \\ \Big\downarrow{\scriptstyle \psi_i} & & \\ P_{i-1} & & \end{array}$$

Here ψ_i (resp. ϕ_i) is a blow up of P_{i-1} (resp. P_i) with smooth centre B_i (resp. A_i) with the same exceptional divisor E_i. The centres of the blow up, B_i and A_i, admit explicit descriptions as projective bundles over the i-th symmetric power, S^iC, of C.

(c) Finally, the variety $P := P_i$ ($i = \frac{d-1}{2}$) can be identified with $\mathbf{P}(\mathcal{W})$.

In order to compute the numbers $h^i(M, \mathrm{ad}(\mathcal{W}))$; $0 \leq i \leq 2$, we remark that it suffices to compute $h^i(P, T_P)$; where T_P denotes the tangent bundle of P. These two numbers can be easily related by an application of Theorem 1.1. But, since the codimension of A_i in P_{i-1} (resp. B_i in P_i) is at least five for $i > 2$, using a Hartog type argument it suffices to compute $h^i(\tilde{P}_2, T_{\tilde{P}_2})$. Now, by an argument very similar in spirit to step 3 in the proof of Theorem 1.4, we reduce to computing $h^i(\tilde{P}_2 - E_2, T_{\tilde{P}_2})$—which is equal to $h^i(P_1, T_{P_1})$ as the map ψ_1 is an isomorphism outside of E_2 and the codimension of B_1 in P_1 is large (cf. [T]). This number can in fact be computed for all values of i