ABELIAN FUNCTIONS

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Abel's theorem and the allied theory of theta functions

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CONTENTS.

CHAPTER I.

THE SUBJECT OF INVESTIGATION.

CHAPTER II.

THE FUNDAMENTAL FUNCTIONS ON A RIEMANN SURFACE.

CHAPTER III.

THE INFINITIES OF RATIONAL FUNCTIONS.

CHAPTER IV.

Specification of a general form of Riemann's integrals.

CHAPTER V.

Certain forms of the fundamental equation of the Riemann surface.

CHAPTER VI.

Geometrical investigations.

CHAPTER VII.

Coordination of simple elements. Transcendental uniform functions.

CHAPTER VIII.

ABEL'S THEOREM. ABEL'S DIFFERENTIAL EQUATIONS.

CHAPTER IX.

JACOBI'S INVERSION PROBLEM.

CHAPTER X.

RIEMANN'S THETA FUNCTIONS. GENERAL THEORY.

CHAPTER XI.

THE HYPERELLIPTIC CASE OF RIEMANN'S THETA FUNCTIONS.

CHAPTER XII.

A PARTICULAR FORM OF FUNDAMENTAL SURFACE.

CHAPTER XIII.

RADICAL FUNCTIONS.

CHAPTER XIV.

FACTORIAL FUNCTIONS.

CHAPTER XV.

RELATIONS CONNECTING PRODUCTS OF THETA FUNCTIONS—INTRODUCTORY.

CHAPTER XVI.

A DIRECT METHOD OF OBTAINING THE EQUATIONS CONNECTING THETA PRODUCTS.

CHAPTER XVII.

THETA RELATIONS ASSOCIATED WITH CERTAIN GROUPS OF CHARACTERISTICS.

CHAPTER XVIII.

TRANSFORMATION OF PERIODS, ESPECIALLY LINEAR TRANSFORMATION.

CHAPTER XIX.

ON SYSTEMS OF PERIODS AND ON GENERAL JACOBIAN FUNCTIONS.

CHAPTER XX.

TRANSFORMATION OF THETA FUNCTIONS.

CHAPTER XXI.

COMPLEX MULTIPLICATION OF THETA FUNCTIONS. CORRESPONDENCE OF POINTS ON A RIEMANN SURFACE.

CHAPTER XXII.

DEGENERATE ABELIAN INTEGRALS.

APPENDIX I.

On algebraic curves in space.

APPENDIX II.

On matrices.

Foreword

Classical algebraic geometry, inseparably connected with the names of Abel, Riemann, Weierstrass, Poincaré, Clebsch, Jacobi and other outstanding mathematicians of the last century, has mainly been an analytical theory. In our century it has been enriched by the methods and ideas of topology and commutative algebra and has the authority of one of the most fundamental mathematical disciplines.

The traditional eclecticism (in the best sense of the word) of algebraic geometry has always been a source of its numerous applications to other branches of mathematics. The role of algebraic geometry as "an applied science" has grown immensely in the last 15–20 years, when its new applications to the problems of non-linear equations and quantum field theory were found.

Mechanics, mathematical and theoretical physics can be called "new" spheres of the application of algebraic geometry. These areas are nontraditional only for the algebraic geometry of the second third of our century, the period when the abstract language of Grothendieck's schemes seemed to replace once and for all the somewhat naive language of classical algebraic geometry.

The results of recent years, amongst which we should specially mention the solution of the Riemann–Schottky problem and the applications of topological gravity to the intersection theory of moduli spaces of algebraic curves, show that now, as in the last century, the relationship between algebraic geometry and physics is by no means a one-way street.

The sudden growth in the number of scholars for whom algebraic geometry has become a working tool has highlighted the lack of relevant mathematical literature. Practically all the books on this subject that are available to the reader of today have been written in abstract algebraic language. The idea of maximum generality inherent in the theory of schemes hampers the reader willing to be quickly introduced to the subject, especially if the reader is a physicist.

It would be an exaggeration to state that suitable literature is completely lacking. The neo-classical style is quite evident in some books of recent decades. Amongst them are *Principles of algebraic geometry* by Griffiths and Harris, *Tata lectures on Theta* by Mumford, and *Theta-functions* by Fay.

Undoubtedly, Baker's book assumes a special place in the list, one guaranteed by the mere fact that its first edition is dated at the end of the last century. But that is not its only special feature. It is surprisingly up-to-date and moreover contains some results which have until now remained beyond the scope of present-day textbooks. It is noteworthy that these particular results are closely related to the above-mentioned applications of algebraic geometry to modern mathematical physics.

With all the variety of results obtained within the framework of classical algebraic geometry, its core consists of relatively few basic definitions and theorems. The list includes the Riemann–Roch theorem, the notion of Jacobian variety of an algebraic curve, Abel's theorem and Jacobi's solution of the inversion problem with the help of theta-functions.

Though the author of the book put in its title only Abel's theorem and the theory of theta-functions, the modest words "and the allied theory" mean "all the rest". This "all the rest" includes, besides the theorems above, elements of uniformization theory of algebraic curves and related theory of automorphic forms and the Schottky model of algebraic curves. Of special importance for modern applications are the sections devoted to the "factorial functions".

It is necessary to emphasize another, and possibly the principal, merit of this book. It exhibits the characteristic feature of classical algebraic geometry — the wish to express the final results in exact analytical formulae. This presupposes the definition of a minimal set of new transcendental functions — the "bricks" of which the whole building can be constructed.

In order to demonstrate it we shall briefly present the key points of the so-called finite-gap (or algebraic–geometrical) integration theory of non-linear equations. At the same time this will allow us to do justice to the author of this book, whose name is perpetuated in the "Baker–Akhiezer" function, the concept of which plays a crucial role in various modern applications of algebraic geometry to non-linear physics.

The algebraic–geometrical integration scheme of non-linear equations

is applicable to all the equations that are considered in the framework of the inverse problem method. Among them are:
the Korteweg-de Vries (KdV) equation

$$u_t - \frac{3}{2}uu_x + \frac{1}{4}u_{xxx} = 0, \tag{0.1}$$

its two-dimensional generalization *the Kadomtsev–Petviashvili (KP) equation*

$$\frac{3}{4}u_{yy} = (u_t - \frac{3}{2}uu_x + \frac{1}{4}u_{xxx})_x, \tag{0.2}$$

the non-linear Schrödinger equation

$$i\psi_t = \psi_{xx} + |\psi|^2\psi = 0, \tag{0.3}$$

the sine–gordon equation

$$u_{tt} - u_{xx} = \sin u \tag{0.4}$$

and many other fundamental equations of modern mathematical physics.

All equations that are considered in the framework of the inverse problem can be represented as compatibility conditions for an over-determined system of auxiliary linear problems.

For example, for the KdV equation (0.1), this system has the form

$$L\psi = 0, \quad \partial_t\psi = A\psi, \tag{0.5}$$

where L and A are

$$L = -\partial_x^2 + u(x,t), \quad A = \partial_x^3 - \frac{3}{2}u\partial_x - \frac{3}{4}u_x. \tag{0.6}$$

The compatibility of (0.5) implies

$$[\partial_t - A, L] = 0 \Longleftrightarrow L_t = [A, L]. \tag{0.7}$$

The operator equation (0.7) is called *the Lax equation.* A wide class of non-linear equations can be represented in the form (0.7), where L and A are ordinary differential equations in the variable x with matrix or scalar coefficients that are functions of the variables x, t:

$$L = \sum_{i=1}^{n} u_i(x,t)\partial_x^i, \quad A = \sum_{i=1}^{m} v_i(x,t)\partial_x^i. \tag{0.8}$$

Each Lax equation is an infinite-dimensional analogue of the completely integrable systems. In particular, it can be included in a hierarchy of an infinite set of commuting flows. For the KdV equation they have the form

$$\partial_n u = f_n(u, u_x, \ldots, u^{(2n+1)}), \quad u = u(x, t, t_3, t_4, \ldots), \quad \partial_n = \frac{\partial}{\partial t_n}, \quad (0.9)$$

and are equivalent to the operator equation

$$\partial_n L = [A_{2n+1}, L], \quad (0.10)$$

where L is the Schrödinger operator and A_{2n+1} is a differential operator of order $2n+1$.

The initial definition of n-gap solutions of the KdV equation was proposed by Novikov who considered the restriction of the KdV equation to the space of stationary solutions of (0.10)

$$f_n(u, u_x, \ldots, u^{(2n+1)}) = 0 \Longleftrightarrow [L, A_{2n+1}] = 0. \quad (0.11)$$

The operator equation (0.11) is a particular case of the more general problem of the classification of commuting ordinary differential operators L_n and L_m of orders n and m, respectively. As a purely algebraic problem it was considered and partly solved in the remarkable works of Burchnall and Chaundy [1], [2] in the 1920s. They proved that for any pair of such operators there exists a polynomial $R(\lambda, \mu)$ in two variables such that

$$R(L_n, L_m) = 0. \quad (0.12)$$

If the orders n and m of these operators are coprime, $(n, m) = 1$, then for each point $Q = (\lambda, \mu)$ of the curve Γ, that is defined in C^2 by the equation $R(\lambda, \mu) = 0$, there corresponds a unique (up to a constant factor) common eigenfunction $\psi(x, Q)$ of L_n and L_m:

$$L_n\psi(x, Q) = \lambda\psi(x, Q); \quad L_m\psi(x, Q) = \mu\psi(x, Q). \quad (0.13)$$

The logarithmic derivative $\psi_x\psi^{-1}$ is a meromorphic function on Γ. In the general position (when Γ is smooth) it has g poles $\gamma_1(x), \ldots, \gamma_g(x)$ in the affine part of the curve, where g is the genus of Γ. The commuting operators L_n, L_m (in this case of coprime orders) are uniquely defined by the polynomial R and by a set of g points $\gamma_1(x_0), \ldots, \gamma_g(x_0)$ on Γ.

In such a form, the solution of the problem is one of pure classification: one set is equivalent to the other. Even the attempt to obtain exact formulae for the coefficients of commuting operators had not been made. Baker proposed making the programme effective by pointing out that the eigenfunction ψ has analytical properties that were introduced by Clebsch, Gordan and himself as a proper generalization of the notion of exponential functions on Riemann surfaces.

The Baker program was rejected by the authors of [1], [2] consciously (see the postscript of Baker's paper [3]) and all these results were forgotten for a long time.

Briefly, the key points of a proof of the above results are the following. The commutativity of L_n and L_m implies that the space $\mathcal{L}(\lambda)$ of solutions of the equation

$$L_n y(x) = \lambda y(x) \tag{0.14}$$

is invariant with respect to the operator L_m. The matrix elements L_m^{ij}, $i,j = 0,\ldots,n-1$, of the corresponding finite-dimensional operator $L_m(\lambda)$,

$$L_m|_{\mathcal{L}(\lambda)} = L_m(\lambda): \ \mathcal{L}(\lambda) \longmapsto \mathcal{L}(\lambda) \tag{0.15}$$

in the canonical basis

$$c_i(x,\lambda,x_0) \in \mathcal{L}(\lambda), \quad c_i(x,\lambda,x_0)|_{x=x_0} = \delta_{ij}, \tag{0.16}$$

are polynomial functions in the variable λ. They depend on the choice of the normalization point $x = x_0$, i.e. $L_m^{ij} = L_m^{ij}(\lambda, x_0)$. The characteristic polynomial

$$R(\lambda,\mu) = \det(\mu - L_m^{ij}(\lambda, x_0)) \tag{0.17}$$

is a polynomial in both variables λ and μ and does not depend on x_0.

According to the property of characteristic polynomials we have

$$R(L_n, L_m) y(x,\lambda) = 0. \tag{0.18}$$

$R(L_n, L_m)$ is an ordinary differential operator. Therefore, if it is not equal to zero then its kernel is finite-dimensional. Hence, (0.18) implies (0.12), and the first statement of [1],[2] is proved.

The equation

$$R(\lambda,\mu) = 0 \tag{0.19}$$

defines the affine part of the algebraic curve Γ in C^2.

Surprisingly, the presentation of the contents in the present book is "parallel" to the solution of this problem. In the first lines we read: *This book is concerned with a particular development of the theory of the algebraic irrationality when a quantity y is defined in terms of a quantity x by mean of an equation of the form*

$$a_0 y^n + a_1 y^{n-1} + \cdots + a_{n-1} y + a_n = 0. \tag{0.20}$$

Possibly, this formulation and the consequent detailed discussion of "what is a *place*" of a Riemann surface, looks too naive for a modern reader, but it has its own advantages, because it allows us to touch the main subjects of the theory right from the start. The general structure of the book is from the particular definitions to a general structure and back to specific problems. For example, in the first chapter of the book just after the definition of algebraic irrationality (0.20) the notion of their rational equivalence is introduced. The invariance of the genus (deficiency) of irrationalities with respect to rational transformations is proved. At the end of this chapter it is proved that "the greatest number of irremovable parameters" for algebraic irrationalities of genus g is equal to $3g-3$ (in modern language this number is the dimension of the moduli space of genus g algebraic curves). And all this in only 13 pages!

Let us return to the classification problem of commuting ordinary differential operators. The consideration of the asymptotic behaviour of the algebraic equation (0.19) in a neighbourhood of "infinity" ($\lambda \to \infty$) allows one to prove that if the orders n and m of L_n and L_m are coprime, then the affine curve (0.19) is compactified by one smooth point, in a neighbourhood of which $\lambda^{-1/n}$ is a local co-ordinate, i.e. infinity is an n-fold branching point of Γ. Hence, equation (0.19) for a generic value of λ has n distinct roots and for each point $Q = (\lambda, \mu)$ of Γ there exists a unique eigenvector $h(Q) = (h_1(Q), \ldots, h_n(Q))$ of the operator $L_m(\lambda)$,

$$L_m(\lambda)h(Q) = \mu h(Q), \qquad (0.21)$$

normalized by the condition $h_0(Q) = 1$. All the other components h_i of this vector are rational functions in λ and μ, i.e. meromorphic functions on Γ. They depend on the choice of normalizing point x_0, $h_i = h_i(Q, x_0)$. In the affine part of Γ the poles of h coincide with zeros on the curve Γ of the minor $L_m^{ij}(\lambda)$, $i, j = 1, \ldots, n-1$. If the curve is smooth then the number of poles is equal to the genus of Γ. The poles $\gamma_1(x_0), \ldots, \gamma_g(x_0)$ depend on x_0.

The common eigenfunction $\psi(x, Q)$ of L_n and L_m is defined up to a constant factor, therefore its logarithmic derivative $\psi_x \psi^{-1}$ is defined uniquely. It follows from the definition of the canonical basis (0.16) that

$$\psi_x(x, Q)\psi^{-1}(x, Q)|_{x=x_0} = h_1(Q, x_0). \qquad (0.22)$$

That proves the second statement of [1],[2]. In order to prove the final statement that the coefficients of R and the divisor $\gamma_s(x_0)$ on the corresponding curve uniquely define the commuting operators, let us consider the analytical properties of $h_1(Q, x_0)$ on Γ, including the infinite point P_0. It turns out that besides the poles $\gamma_s(x_0)$ in the affine part of Γ it has a simple pole of the form

$$h_1(Q, x_0) = k + O(k^{-1}), \quad k^n = \lambda, \quad Q = (\lambda, \mu), \tag{0.23}$$

at infinity.

Let Γ be a smooth genus g algebraic curve with a local co-ordinate $k^{-1}(Q)$ in a neighbourhood of a puncture P_0. Then according to the Riemann–Roch theorem for a generic set of g points γ_s there exists a unique function h_1 that has at most simple poles at these points and has the form (0.23) in the neighbourhood of P_0.

The proof of this particular case of the Riemann–Roch theorem can be found in Chapter IV of the book. In the book the corresponding function is sometimes called a "Weierstrass function". It is one of the fundamental rational functions through which all the other rational functions can be expressed.

Let $z_s(Q)$ be a local co-ordinate near the point γ_s, then the corresponding function has an expansion

$$h_1(Q) = \frac{a_s}{z_s(Q) - z_s(\gamma_s)} + O(1). \tag{0.24}$$

The coefficients a_s of the expansion (0.24) are uniquely defined by the set $\gamma_1, \ldots, \gamma_g$, i.e. $a_s = a_s(\gamma_1, \ldots, \gamma_g)$.

The common eigenfunction $\psi(x, Q, x_0)$ of the operators L_n, L_m which is normalized by the condition $\psi(x = x_0, Q, x_0) = 1$ is equal to

$$\psi(x, Q, x_0) = \sum_{i=0}^{n-1} h_i(Q, x_0) c_i(x, \lambda, x_0). \tag{0.25}$$

The functions c_i are entire functions of the variable λ. Hence, ψ is a meromorphic function on Γ except for infinity. It has poles at $\gamma_s(x_0)$ and g zeros $\gamma_s(x)$ that are poles of its logarithmic derivative. In their vicinity we have

$$\psi_x \psi^{-1} = \frac{\partial_x z_s(\gamma_s(x))}{z_s(Q) - z_s(\gamma_s)} + O(1). \tag{0.26}$$

The comparison of (0.24) and (0.26) implies that

$$\partial_x z_s(\gamma_s(x)) = a_s(\gamma_1(x), \ldots, \gamma_g(x)). \tag{0.27}$$

Equations (0.27) consist of a well-defined system of differential equations of the first order. A solution of this system is defined by the initial data $\gamma_1(x_0), \ldots, \gamma_g(x_0)$. That proves the final statement of Burchnall and Chaundy.

In [3] it was proposed to consider the analytical properties of the common eigenfunction ψ on the compactified curve Γ. From the purely algebraic point of view it is a "forbidden" function, because it has an essential singularity at the infinite point P_0. But this essential singularity is of a very special form — it is of exponential type. It follows from (0.23) that

$$\begin{aligned}\psi(x, Q, x_0) &= \exp(\int_{x_0}^{x} h_1(Q, x)dx) \\ &= e^{k(x-x_0)}(1 + \sum_{s=1}^{\infty} \xi_s(x, x_0)k^{-s}), \ \lambda = k^n(Q) \to \infty.\end{aligned} \tag{0.28}$$

The theory of such functions, considered as a natural generalization of the exponential function to Riemann surfaces, is very deeply connected with the theory of the so-called *factorial* functions that can be found in Chapter XIV of the book. These functions are single-valued on the surface dissected along cycles and their values on different sides of the cuts satisfy special boundary conditions. In modern language they are solutions of the Riemann–Hilbert problem on a Riemann surface. The expression of such functions in terms of Riemann theta-functions is one of the main goals of the chapter.

Baker pointed out that these results should make it possible to find the exact formulae for the coefficients of the commuting operators of coprime orders. It turned out that this program was realized only in [4],[5] (though at that time the author was not aware of the remarkable results of Burchnall, Chaundy and Baker) where the commuting pairs of ordinary differential operators were considered in connection with the problem of constructing solutions to the KP equation.

The common eigenfunction of commuting operators is a particular case of the general definition of scalar the *multi-point* and *multi-variable* Clebsch–Gordan–Baker–Akhiezer function (or more simply the Baker–Akhiezer function). Let Γ be a non-singular algebraic curve of genus g with N punctures P_α and fixed local parameters $k_\alpha^{-1}(Q)$ in neighbourhoods of these punctures. For any set of points $\gamma_1, \ldots, \gamma_g$ in general position, there exists a unique (up to constant factor, $c(t_{\alpha,i})$) function $\psi(t, Q)$, $t = (t_{\alpha,i})$, $\alpha = 1, \ldots, N$, $i = 1, \ldots,$ such that:

(i) the function ψ (as a function of the variable $Q \in \Gamma$) is meromorphic everywhere except for the points P_α and it has at most simple poles at the points $\gamma_1, \ldots, \gamma_g$ (if all of them are distinct);

(ii) in a neighbourhood of the point P_α the function ψ has the form

$$\psi(t,Q) = \exp\Big(\sum_{i=1}^{\infty} t_{\alpha,i}k_\alpha^i\Big)\Big(\sum_{s=0}^{\infty}\xi_{s,\alpha}(t)k_\alpha^{-s}\Big),\, k_\alpha = k_\alpha(Q). \tag{0.29}$$

We see that the Baker–Akhiezer function ψ depends on the variables $t = \{t_{1,i},\dots,t_{n,i}\}$ and on external parameters.

From the uniqueness of the Baker–Akhiezer function it follows that for each pair (α,n) there exists a unique operator $L_{\alpha,n}$ of the form

$$L_{\alpha,n} = \partial_{\alpha,1}^n + \sum_{j=1}^{n-1} u_j^{(\alpha,n)}(t)\partial_{\alpha,1}^j \tag{0.30}$$

(where $\partial_{\alpha,i} = \partial/\partial t_{\alpha,i}$) such that

$$(\partial_{\alpha,i} - L_{\alpha,n})\psi(t,Q) = 0. \tag{0.31}$$

The idea of the proof of theorems of this type which was proposed in [4] is universal.

For any formal series of the form (0.29) there exists a unique operator $L_{\alpha,n}$ of the form (0.30) such that

$$(\partial_{\alpha,i} - L_{\alpha,n})\psi(t,Q) = O(k^{-1})\exp\Big(\sum_{i=1}^{\infty} t_{\alpha,i}k_\alpha^i\Big). \tag{0.32}$$

The coefficients of $L_{\alpha,n}$ are differential polynomials with respect to $\xi_{s,\alpha}$. They can be found after substitution of the series (0.29) into (0.32).

It turns out that, if the series (0.29) is not formal but is an expansion of the Baker–Akhiezer function in the neighbourhood of P_α, then the congruence (0.32) becomes an equality. Indeed, let us consider the function

$$\psi_1 = (\partial_{\alpha,n} - L_{\alpha,n})\psi(t,Q). \tag{0.33}$$

It has the same analytical properties as ψ, except for one. The expansion of this function in a neighbourhood of P_α starts from $O(k^{-1})$. From the uniqueness of the Baker–Akhiezer function it follows that $\psi_1 = 0$ and the equality (0.31) is proved.

A corollary is that the operators $L_{\alpha,n}$ satisfy the compatibility conditions

$$[\partial_{\alpha,n} - L_{\alpha,n}, \partial_{\alpha,m} - L_{\alpha,m}] = 0. \tag{0.34}$$

The equations (0.34) are gauge invariant. For any function $g(t)$ operators

$$\tilde{L}_{\alpha,n} = gL_{\alpha,n}g^{-1} + (\partial_{\alpha,n}g)g^{(-1)} \tag{0.35}$$

have the same form (0.30) and satisfy the same operator equations (0.34). The gauge transformation (0.35) corresponds to the gauge transformation of the Baker–Akhiezer function

$$\psi_1(t,Q) = g(t)\psi(t,Q) \tag{0.36}$$

In the one-point case the Baker–Akhiezer function has an exponential singularity at a single point P_1 and depends on a single set of variables. Let us choose the normalisation of the Baker–Akhiezer function with the help of the condition $\xi_{1,0} = 1$, i.e. an expansion of ψ in the neighbourhood of P_1 equals

$$\psi(t_1, t_2, \ldots, Q) = \exp\Big(\sum_{i=1}^{\infty} t_i k^i\Big)\Big(\sum_{s=0}^{\infty} \xi_s(t) k^{-s}\Big). \tag{0.37}$$

In this case the operator L_n has the form

$$L_n = \partial_1^n + \sum_{i=0}^{n-2} u_i^{(n)} \partial_1^i. \tag{0.38}$$

If we denote t_1, t_2, t_3 by x, y, t, respectively, then from (0.34) it follows (for $n = 2$, $m = 3$) that $u(x, y, t, t_4, \ldots)$ satisfies the KP equation (0.2). The exact formula for these solutions in terms of the Riemann theta-function is based on the exact formula for the Baker–Akhiezer function.

Let us fix the basis of cycles a_i, b_i, $i = 1, \ldots, g$ on Γ with the canonical matrix of intersections: $a_i \circ a_j = b_i \circ b_j = 0$, $a_i \circ b_j = \delta_{ij}$. The basis of normalized holomorphic differentials $\omega_j(Q)$, $j = 1, \ldots, g$ is defined by conditions

$$\oint_{a_i} \omega_j = \delta_{ij}. \tag{0.39}$$

The b-periods of these differentials define the so-called Riemann matrix

$$B_{kj} = \oint_{b_j} \omega_k. \tag{0.40}$$

The basic vectors e_k of C^g and the vectors B_k, which are the columns of matrix (0.40), generate a lattice $\mathcal{B}$ in C^g. The g-dimensional complex torus

$$J(\Gamma) = C^g/\mathcal{B}, \quad \mathcal{B} = \sum n_k e_k + m_k B_k, \quad n_k, m_k \in Z, \tag{0.41}$$

is called the Jacobian variety of Γ. A vector with co-ordinates

$$A_k(Q) = \int_{q_0}^{Q} \omega_k \tag{0.42}$$

defines the Abel map

$$A: \quad \Gamma \longmapsto J(\Gamma) \tag{0.43}$$

which depends on the choice of the initial point q_0.

The Riemann matrix has a positive-definite imaginary part. The entire function of g variables

$$\theta(z) = \theta(z|B) = \sum_{m \in Z^g} e^{2\pi i(z,m)+\pi i(Bm,m)}, \tag{0.44}$$

$$z = (z_1, \ldots, z_n), m = (m_1, \ldots, m_n), (z, m) = z_1 m_1 + \ldots + z_n m_n,$$

is called the Riemann theta-function. It has the following monodromy properties;

$$\theta(z + e_k) = \theta(z), \quad \theta(z + B_k) = e^{-2\pi i z_k - \pi i B_{kk}} \theta(z). \tag{0.45}$$

The function $\theta(A(Q) - Z)$ is a multi-valued function of Q. But according to (0.45), the zeros of this function are well-defined. For Z in a general position the equation

$$\theta(A(Q) - Z) = 0 \tag{0.46}$$

has g zeros $\gamma_1, \ldots, \gamma_g$. The vector Z and the divisor of these zeros are connected by the relation

$$Z_k = \sum_{i=1}^{g} A(\gamma_s) + \mathcal{K}, \tag{0.47}$$

where $\mathcal{K}$ is the vector of Riemann constants.

Let us introduce the normalized Abelian differentials $d\Omega_{\alpha,i}$ of the second kind. The differential $d\Omega_{\alpha,i}$ is holomorphic on Γ except for the puncture P_α. In the neighbourhood of this point it has the form

$$d\Omega_{\alpha,i} = d(k_\alpha^i + O(1)). \tag{0.48}$$

"Normalized" means that it has zero a-periods

$$\oint_{a_j} d\Omega_{\alpha,i} = 0. \tag{0.49}$$

Consider the function

$$\mathcal{E}(t, Q) = \exp\big(\sum_{\alpha,j} t_{\alpha,j} \int_{q_0}^{Q} d\Omega_{\alpha,j}\big). \tag{0.50}$$

It has the same exponential singularities of the form (0.29) at the punctures as the Baker–Akhiezer function, but it is a single-valued function on Γ dissected along a-cycles only. Its values on two sides of the cycle a_i differ by the factor

$$e^{2\pi i U_i} = \exp\big(\sum_{\alpha,j} t_{\alpha,j} U^i_{\alpha,j}\big), \qquad (0.51)$$

where

$$U^i_{\alpha,j} = \frac{1}{2\pi i} \oint_{b_i} d\Omega_{\alpha,j}. \qquad (0.52)$$

Consider the function

$$\phi(V,Q) = \frac{\theta(A(Q)+V-Z)}{\theta(A(Q)-Z)}, \qquad (0.53)$$

where V is a vector with co-ordinates $V_1,\ldots,V_g$. This function is meromorphic on Γ dissected along a-cycles and has g poles (depending on Z). It follows from the monodromy properties (0.45), that the boundary values of ϕ on two sides of the a_j cycles satisfy the relation

$$\phi^+ = e^{-2\pi i V_j}\phi^-. \qquad (0.54)$$

Such multi-valued functions are called "factorial" functions in the book.

Equalities (0.50-0.54) imply that the function

$$\psi(t,Q) = \mathcal{E}(t,Q)\frac{\theta(A(Q)+\sum_{\alpha,j} t_{\alpha,j}U_{\alpha,j} - Z)}{\theta(A(Q)-Z)} \qquad (0.55)$$

is a single-valued function on Γ and has all the other properties of the desired function. Therefore, the existence of the Baker–Akhiezer function is proved. Let $\tilde{\psi}$ be any function with the same analytical properties. The ratio $\tilde{\psi}/\psi$ is a meromorphic function with at most g poles. The Riemann–Roch theorem implies that such a function is equal to a constant. Hence, the uniqueness of the Baker–Akhiezer function (up to a constant factor) is also proved.

The coefficients of the operators $L_{\alpha,j}$ which are defined by the equations (0.31) are differential polynomials in the coefficients of the expansions of the second factor in (0.55) near the punctures. Hence, they can be expressed as differential polynomials in terms of Riemann theta-functions. For example, the algebraic–geometrical solutions of the KP hierarchy have the form

$$u(x,y,t,t_4,\ldots) = 2\partial_x^2 \ln\theta(xU_1+yU_2+tU_3+\cdots+Z)+const. \qquad (0.56)$$

The common eigenfunction of commuting operators of coprime orders is the particular case of a one-point Baker–Akhiezer function corresponding to $t_1 = x, t_2 = 0, t_3 = 0, \ldots$. Therefore, the coefficients of such operators (in general position) are differential polynomials in terms of the Riemann theta-functions. This has an important corollary. The coefficients of commuting differential operators of coprime orders are meromorphic functions of the variable x. Moreover, in general position they are quasi-periodic functions of x. The last statement presents evidence that the theory of commuting operators is connected with the spectral Floquet theory of periodic differential operators. These connections were missing in [1], [2], [3].

The origin of Riemann surfaces in the spectral theory of ordinary periodic differential operators appears now to be self-evident. Indeed, for such an operator the space $\mathcal{L}(\lambda)$ of solutions of equation (0.14) is invariant with respect to the monodromy operator

$$\hat{T} : y(x) \longmapsto y(x+T). \tag{0.57}$$

Let $T(\lambda)$ be the corresponding finite-dimensional operator. The characteristic equation

$$R(w, \lambda) = \det(w - T(\lambda)) = 0 \tag{0.58}$$

defines the Riemann surface of Bloch solutions; the common eigenfunctions for the operator L and monodromy operator, i.e.

$$L\psi(x, Q) = \lambda\psi(x, Q), \quad \psi(x+T, Q) = w\psi(x, Q), \; Q = (w, \lambda). \tag{0.59}$$

For the general periodic operator the Riemann surface of Bloch solutions has an infinite genus. Periodic operators for which this surface has a finite genus are operators that commute with some other ordinary differential operators. In this case the Riemann surface of Bloch solutions and the algebraic curve of common eigenfunctions of commuting ordinary differential operators are isomorphic.

Originally, the classification problem of commuting ordinary differential operators was posed for operators of arbitrary orders. In [1],[2] it was mentioned that there are no approaches to the solution of this problem if the orders of these operators are not coprime. The complete solution of the problem was obtained in [6]. It turned out that such operators are defined uniquely by the polynomial $R(\lambda, \mu)$ (0.17), by a vector bundle over Γ of rank r and degree rg and by a set of $r-1$ arbitrary functions $w_0(x), \ldots, w_{r-2}(x)$. Here r is a common divisor of the orders n, m. It equals the number of linear independent solutions of the equations (0.13). The problem of reconstruction of the coefficients of commuting differential operators of the rank $r > 1$ is reduced to the system of linear integral equations and is beyond the framework of the book.

To conclude let us give the list of reviews of finite-gap theory [7]-[11]. We would like to specially mention the work [12], where it was proved that the function that is given by formula (0.56) is a solution of the KP equation if and only if the matrix B that defines the theta-function is the Riemann matrix of some algebraic curve. This statement solves the Schottky problem and was conjectured by Novikov.

It would be fair to say that the most exciting results of recent years in algebraic geometry and in mathematical physics are connected with the application of so-called topological field theories, matrix integrals to the intersection theory on the moduli spaces of algebraic curves with punctures [13]-[14].

There is no doubt that this book will provide an excellent introduction to the algebraic–geometrical techniques that are necessary for those who are interested in this field.

We wish success to those who now begin to turn the pages which follow.

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PREFACE.

It may perhaps be fairly stated that no better guide can be found to the analytical developments of Pure Mathematics during the last seventy years than a study of the problems presented by the subject whereof this volume treats. This book is published in the hope that it may be found worthy to form the basis for such study. It is also hoped that the book may be serviceable to those who use it for a first introduction to the subject. And an endeavour has been made to point out what are conceived to be the most artistic ways of formally developing the theory regarded as complete.

The matter is arranged primarily with a view to obtaining perfectly general, and not merely illustrative, theorems, in an order in which they can be immediately utilised for the subsequent theory; particular results, however interesting, or important in special applications, which are not an integral portion of the continuous argument of the book, are introduced only so far as they appeared necessary to explain the general results, mainly in the examples, or are postponed, or are excluded altogether. The sequence and scope of ideas to which this has led will be clear from an examination of the table of Contents.

The methods of Riemann, as far as they are explained in books on the general theory of functions, are provisionally regarded as fundamental; but precise references are given for all results assumed, and great pains have been taken, in the theory of algebraic functions and their integrals, and in the analytic theory of theta functions, to provide for alternative developments of the theory. If it is desired to dispense with Riemann's existence theorems, the theory of algebraic functions may be founded either on the arithmetical ideas introduced by Kronecker and by Dedekind and Weber; or on the quasi-geometrical ideas associated with the theory of adjoint polynomials; while in any case it does not appear to be convenient to avoid reference to either class of ideas. It is believed that, save for some points in the periodicity of Abelian integrals, all that is necessary to the former elementary development will be found in Chapters IV. and VII., in connection with which the reader may consult the recent paper of Hensel, *Acta Mathematica*, XVIII. (1894), and also the papers of Kronecker and of