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Complex Algebraic Surfaces

Second Edition

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INTRODUCTION

This book is a modified version of a course given at Orsay in 1976–77. The aim of the course was to give a comparatively elementary proof of the Enriques classification of complex algebraic surfaces, accessible to a student familiar with the basic language of algebraic geometry (divisors, differential forms, ...) as well as sheaf cohomology. I have, however, preferred to assume along the way various hard theorems from algebraic geometry, rather than resort to complicated and artificial proofs.

Here is an outline of the course. The first two chapters introduce the basic tools for the study of surfaces: in Chapter I we define the intersection form on the Picard group, and establish its properties; assuming the duality theorem we deduce the fundamental results (the Riemann–Roch theorem, the genus formula). Chapter II is devoted to the structure of birational maps; we show in particular that every surface is obtained from a minimal surface by a finite number of blow-ups. The chapter ends with Castelnuovo's contractibility criterion, which characterizes exceptional curves by their numerical properties.

The classification begins in Chapter III with ruled surfaces, that is, surfaces birational to $\mathbb{P}^1 \times C$. We show that (except in the rational case) their minimal models are $\mathbb{P}^1$-bundles over a base curve C, and we study their geometry. Chapter IV gives some examples of rational surfaces; we take a stroll through the huge menagerie collected by the geometers of the 19th century (the Veronese surface, del Pezzo surfaces, ...).

The next two chapters are perhaps the keystone of the classification; they give the characterization of ruled surfaces by their numerical properties – more precisely, by the vanishing of the 'plurigenera' P_n. Surfaces with $q = 0$ are treated in Chapter V, where we prove Castelnuovo's theorem: a surface with $q = P_2 = 0$ is rational. We deduce two important consequences: the structure of minimal rational surfaces and the uniqueness of the minimal model of a non-ruled surface. In Chapter VI we begin the study of surfaces with $q > 0$. We show without too much trouble that a surface having $p_g = 0$ and $q \geqslant 2$ is ruled; which leaves certain non-ruled surfaces with $p_g = 0$ and $q = 1$. According to an idea of Enriques one can classify these surfaces very precisely, and show that they have $P_{12} > 0$. Thus a surface is ruled if and only if $P_{12} = 0$ (Enriques' theorem).

Chapter VII, which is very short, introduces the Kodaira dimension κ, which is a convenient invariant for the classification of surfaces. Ruled surfaces are characterised by $\kappa = -\infty$; the three ensuing chapters deal with surfaces with $\kappa = 0, 1$, and 2. Surfaces with $\kappa = 0$ fall into four classes: $K3$ surfaces, Enriques surfaces, Abelian surfaces, and bielliptic surfaces. The bielliptic surfaces were already listed in Chapter VI, in the context of surfaces with $p_g = 0$ and $q = 1$; here we study $K3$ surfaces and Enriques surfaces, and give numerous examples.

In Chapter IX we show that surfaces with $\kappa = 1$ have a (not necessarily rational) pencil of elliptic curves; conversely we study those surfaces with such a pencil.

Finally Chaper X concerns surfaces with $\kappa = 2$, said to be of general type; although these surfaces are the most general, there is not very much that we can say about them. We have limited ourselves to giving some examples and proving Castelnuovo's inequality $\chi(\mathcal{O}_S) > 0$.

In Appendix A we sketch (without proof) the classification of surfaces in characteristic p, and in Appendix B that of complex compact surfaces. Appendix C indicates some of the new results (or new approaches to old results) which have been obtained since the first appearance of this book.

It is hard to claim any originality in a subject whose main theorems were proved at the turn of the century. I have been largely inspired by the existing literature, in particular by Shafarevich's seminar [Sh 2]; in a historical note at the end of each chapter I have tried to describe the origins of the principal results. The exercises indicate various possible extensions to the course.

NOTATION

By 'surface' we shall mean smooth projective surface over the field $\mathbb{C}$ of complex numbers. Let S be a surface, and D, D' two divisors on S. We write:

$D \equiv D'$ if D and D' are linearly equivalent

$\mathcal{O}_S(D)$: the invertible sheaf corresponding to D

$H^i(S, \mathcal{O}_S(D))$, or simply $H^i(D)$: the ith cohomology group of the sheaf $\mathcal{O}_S(D)$

$h^i(D) = \dim_{\mathbb{C}} H^i(D)$

$\chi(\mathcal{O}_S(D)) = h^0(D) - h^1(D) + h^2(D)$, the Euler–Poincaré characteristic of the sheaf $\mathcal{O}_S(D)$

$|D|$ = the set of effective divisors linearly equivalent to D
= the projective space corresponding to $H^0(D)$

K_S or K = 'the' canonical divisor = a divisor such that $\mathcal{O}_S(K) \cong \Omega_S^2$

Pic S = the group of divisors on S modulo linear equivalence
$\cong$ group of isomorphism classes of invertible sheaves

$NS(S)$ = the Néron–Severi group of S (I.10)

Alb(S) = the Albanese variety of S (see Chapter V)

$q(S)$ or $q = h^1(\mathcal{O}_S) = h^0(\Omega_S^1)$

$p_g(S)$ or $p_g = h^2(\mathcal{O}_S) = h^0(\mathcal{O}_S(K))$

$P_n(S)$ or $P_n = h^0(\mathcal{O}_S(nK))$ (for $n \geqslant 1$)

$b_i(S)$ or $b_i = \dim_{\mathbb{R}} H^i(S, \mathbb{R})$

$\chi_{\text{top}}(S) = \sum (-1)^i b_i(S)$