

These lecture notes are intended as a non-technical overview of scattering theory. The point of view adopted throughout is that scattering theory provides a parametrization of the continuous spectrum of an elliptic operator on a complete manifold with uniform structure at infinity. The simple and fundamental case of the Laplacian on Euclidean space is described in the first two lectures to introduce the basic framework of scattering theory. In the next three lectures various results on Euclidean scattering, and the methods used to prove them, are outlined. In the last three lectures these ideas are extended to non-Euclidean settings.

These lecture notes will be of interest to researchers and graduate students in analysis and differential geometry.

Cambridge University Press
0521498104 - Geometric Scattering Theory
Richard B. Melrose
Frontmatter
[More information](#)

Geometric scattering theory

Cambridge University Press
0521498104 - Geometric Scattering Theory
Richard B. Melrose
Frontmatter
[More information](#)

Geometric scattering theory

RICHARD B. MELROSE
Massachusetts Institute of Technology



Cambridge University Press
0521498104 - Geometric Scattering Theory
Richard B. Melrose
Frontmatter
[More information](#)

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
40 West 20th Street, New York, NY 10011-4211, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

© Cambridge University Press 1995

First Published 1995

Library of Congress cataloging-in-publication data available

A catalogue record for this book is available from the British Library

ISBN 0-521-49673-X hardback
ISBN 0-521-49810-4 paperback

Transferred to digital printing 2003

Contents

<i>List of Illustrations</i>	<i>page</i> viii
<i>Preface</i>	ix
<i>Introduction</i>	xi
1 Euclidean Laplacian	1
1.1 The Laplacian	1
1.2 Spectral resolution	2
1.3 Scattering matrix	4
1.4 Resolvent family	6
1.5 Limiting absorption principle	7
1.6 Analytic continuation	9
1.7 Asymptotic expansion	11
1.8 Radial compactification	13
2 Potential scattering on \mathbb{R}^n	15
2.1 The resolvent of $\Delta + V$	15
2.2 Poles of the resolvent	18
2.3 Boundary pairing	19
2.4 Formal solutions	21
2.5 Unique continuation	21
2.6 Perturbed plane waves	22
2.7 Relative scattering matrix	22
2.8 Asymptotics of the resolvent	24
2.9 L^2 eigenfunctions	25
2.10 Zero energy states	25
2.11 Meromorphy of the scattering matrix	26
3 Inverse scattering	27
3.1 Radon transform	27
3.2 Wave group	30
3.3 Wave operators	33

vi	<i>Contents</i>	
3.4	Lax-Phillips transform	33
3.5	Travelling waves	34
3.6	Near-forward scattering	36
3.7	Constant-energy inverse problem	37
3.8	Exponential solutions	39
3.9	Backscattering	40
4	Trace formulæ and scattering poles	42
4.1	Determinant and scattering phase	43
4.2	Poisson formula	45
4.3	Existence of poles	46
4.4	Lax-Phillips semigroup	47
4.5	Counting function	48
4.6	Pole-free regions	51
5	Obstacle scattering	52
5.1	Obstacles	53
5.2	Scattering operator	55
5.3	Reflected geodesics	56
5.4	Ray relation	59
5.5	Trapped rays	62
6	Scattering metrics	65
6.1	Manifolds with boundary	66
6.2	Hodge theorem	67
6.3	Pseudodifferential operators	68
6.4	Symbol calculus	71
6.5	Index theorem	74
6.6	Limiting absorption principle	75
6.7	Generalized eigenfunctions	76
6.8	Scattering matrix	77
6.9	Long-range potentials	79
6.10	Other theorems?	79
7	Cylindrical ends	80
7.1	b-geometry	81
7.2	Thresholds	83
7.3	Scattering matrix	85
7.4	Boundary expansions and pairing	86
7.5	Hodge theory	87
7.6	Atiyah-Patodi-Singer index theorem	88
7.7	b-Pseudodifferential operators	90
7.8	Trace formula and spectral asymptotics	91
7.9	Manifolds with corners	92

		<i>Contents</i>	vii
8	Hyperbolic metrics		94
8.1	Warped products		94
8.2	Conformally compact manifolds		96
8.3	0-geometry and analysis		98
8.4	The Laplacian		100
8.5	Analytic continuation		102
8.6	Finite volume quotients		103
8.7	hc-geometry		103
8.8	Spectrum		104
	<i>References</i>		106
	<i>Index</i>		114

List of Illustrations

1	The contours $\gamma_+(\lambda')$ and $\gamma_-(\lambda')$.	8
2	Analytic continuation of the resolvent for n odd.	10
3	Analytic continuation of the resolvent for n even.	11
4	Stereographic, or radial, compactification of \mathbb{R}^n .	13
5	Poles of the analytic continuation of $R_V(\lambda)$ (n odd).	20
6	The Lax-Phillips semigroup.	47
7	Reflected geodesics.	57
8	Non-uniqueness of extension of reflected geodesics.	58
9	Two secret rooms.	61
10	The compactified scattering cotangent bundle.	73
11	Geodesic of a scattering metric.	78
12	Spectrum of the Laplacian of an exact b-metric.	84
13	Geodesics for a conformally compact metric.	102

Preface

These notes are based on lectures delivered at Stanford University in January¹ 1994 and then repeated at MIT in the Spring semester. I am very grateful to the members of the Mathematics Department at Stanford, and in particular Ralph Cohen, for the invitation and hospitality. My especial thanks to those who attended the lectures and contributed in one way or another. I am particularly pleased to acknowledge the influence on my thinking of two of the members of the audience, Ralph Phillips and Joe Keller. Rafe Mazzeo encouraged me to write up the lectures, provided me with his own notes and, as if that were not enough, made many helpful comments on the manuscript. I should also like to extend my thanks to Sang Chin, Daniel Grieser, Andrew Hassell, Mark Joshi, Olivier Lafitte, Eckhard Meinrenken, Edith Mooers and Andras Vasy, who attended the second hearing² of the lectures at MIT and together made many useful remarks; Andras Vasy was particularly helpful in reading and correcting the notes as they dribbled out. I would also like to thank Tanya Christiansen and Gunther Uhlmann for their assistance and Lars Hörmander, Georgi Vodev and Maciej Zworski for their comments on later versions of the manuscript.³

It is my hope that these notes may serve as an introduction to an active and growing area of research, although I fear they represent a rather steep learning curve.

¹ It was a horrible month in Cambridge I am told, very pleasant indeed in Palo Alto. This footnote is an indication of things to come in the body of the notes. If you can't stand it, stop now!

² Of course I had really wanted to do things in the other order but did not manage to get my thoughts together in time.

³ Of course, I claim sole credit for all remaining errors.

Introduction

The lectures on which these notes are based were intended as an, essentially non-technical, overview of scattering theory. The point of view adopted throughout is that scattering theory provides a parametrization of the continuous spectrum of an elliptic operator on a complete manifold with uniform structure at infinity. The simple, and fundamental, case of the Laplacian on Euclidean space is described in the first two lectures to introduce the basic framework of scattering theory. In the next three lectures various results on Euclidean scattering, and the methods used to prove them, are outlined. In the last three lectures these ideas are extended to non-Euclidean settings. This is an area of much current research and my idea was to show how similar the Euclidian and the less familiar cases are. Some of the interactions of scattering theory with hyperbolic geometry, index theory and Hodge theory are also indicated.

I have made no attempt at completeness here but simply described what time, and my own tastes, indicate. In particular there should be at least three times as many references as there are. If I have offended by omitting reference to important work, this should not be interpreted as a deliberate slight! In writing up the lectures I have made extensive use of footnotes to cover more subtle points, to clarify statements that were felt to be obscure, by someone, and to make comments. These asides can be freely ignored.