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Harmonic Approximation

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Preface

The year 1885 has a special significance in the history of approximation theory. It was then that Weierstrass published his famous result which says that a continuous function on a closed bounded interval of the real line can be uniformly approximated by polynomials. The same year saw the birth of holomorphic approximation in the celebrated paper of Runge [Run]. Given an open set Ω in the complex plane $\mathbf{C}$, which compact subsets K have the property that any holomorphic function defined on a neighbourhood of K can be uniformly approximated on K by functions holomorphic on Ω? Runge's Theorem supplies the answer: precisely the sets K such that $\Omega \backslash K$ has no components which are relatively compact in Ω. Since Runge's original work holomorphic approximation has developed into a significant research area. We mention particularly the contributions of Carleman [CarT], Alice Roth [Rot1], [Rot3], Mergelyan [Mer], Arakelyan [Ara1] and Nersesyan [Ner]. A helpful account of these and other results can be found in the book by Gaier [Gai]. The purpose of these notes is to give a corresponding account of the theory of harmonic approximation in Euclidean space $\mathbf{R}^n$ $(n \geq 2)$.

The starting point in the history of harmonic approximation is not as easy to identify. In the case of approximation in higher dimensions, the paper of Walsh [Wal] in 1929 seems a reasonable choice, but for approximation in the plane mention must also be made of work of Lebesgue [Leb] in 1907. Which compact sets K in $\mathbf{R}^n$ have the property that any harmonic function defined on a neighbourhood of K can be uniformly approximated on K by harmonic polynomials? Walsh's Theorem tells us that, if $\mathbf{R}^n \backslash K$ is connected, then such approximation is always possible. However, unlike the case of holomorphic approximation, the converse to this statement is false. The characterization of compact sets K with the above approximation property is rather more delicate and involves the potential theoretic notion of "thin sets": denoting by $\widehat{K}$ the union of K with the bounded components of $\mathbf{R}^n \backslash K$, the relevant condition is that $\mathbf{R}^n \backslash \widehat{K}$ and $\mathbf{R}^n \backslash K$ must be thin at the same points of K (see [Gar3]).

The literature on harmonic approximation has also become extensive. There was a period of significant development in the 1940's, due particularly to Keldyš [Kel], Landkof (see [Lan] and the references given there), Brelot [Bre1] and Deny [Den1], [Den2]. Most of this work was concerned with the question of approximation by locally- (rather than by globally-) defined harmonic functions. Which compact sets K in $\mathbf{R}^n$ have the property that any function which is continuous on K and harmonic on the interior K° can be uniformly approximated by functions harmonic on a neighbourhood of K? The answer here also involves thin sets: $\mathbf{R}^n \backslash K$ and $\mathbf{R}^n \backslash K^\circ$ must be thin at the same points.

Until comparatively recently most of the work was in terms of approximation on *compact* sets. This changed in the early 1980's due to two papers by Gauthier, Goldstein and Ow [GGO1], [GGO2]. Inspired by work of Alice Roth [Rot3] in the holomorphic case, they developed a technique of "fusing" two harmonic functions which are close in value on a certain set. As a result they obtained, in particular, a generalization of Walsh's Theorem to the case of approximation on closed (but not necessarily bounded) sets E in $\mathbf{R}^n$: if $(\mathbf{R}^n)^* \backslash E$ is connected and locally connected (where $(\mathbf{R}^n)^*$ denotes the one-point compactification $\mathbf{R}^n \cup \{\infty\}$ of $\mathbf{R}^n$), then functions harmonic on an open set containing E can be uniformly approximated on E by functions harmonic on all of $\mathbf{R}^n$. As was the case with Walsh's Theorem, the above hypotheses concerning connectedness are sufficient, but not necessary for this type of approximation to be possible. A complete characterization of sets E which possess this approximation property has recently been given by the author [Gar3]. Indeed, a recent period of rapid development, due to several authors, has brought a new coherence and substance to the whole subject of harmonic approximation. The purpose of these lecture notes is to give an organised account of harmonic approximation which includes many of these new results.

The plan of these notes is as follows. We assume that the reader is familiar with an introductory text on potential theory such as the book by Helms [Helm], but for convenience we collect together in a preliminary chapter some particularly relevant facts concerning thin sets. Uniform harmonic approximation on compact sets, and then on relatively closed sets, is dealt with in Chapters 1 and 3 respectively. The contents of Chapter 3 and much of the subsequent work rely on a fusion result derived in Chapter 2. We then turn our attention to the question of better-than-uniform approximation: that is, can it be arranged that the error in our approximation decays to 0 as we approach "infinity"? In the holomorphic case one such famous result is due to Carleman [CarT]. He showed that, given any continuous functions $f : \mathbf{R} \to \mathbf{C}$ and $\epsilon : \mathbf{R} \to (0, 1]$, there exists an entire function g such that $|g - f| < \epsilon$ on $\mathbf{R}$. Various results of this type are presented in

Chapters 3-5. Chapter 6 contains analogous and related results concerning the approximation and extension of superharmonic functions.

Finally, one of the most rewarding aspects of the whole subject is its potential for applications, sometimes surprising ones. A selection of these applications will be presented in Chapters 7 and 8. We mention two of them here briefly. The first, which appears in Chapter 7, concerns the Dirichlet problem on an unbounded open set Ω in $\mathbf{R}^n$. Given any continuous function f on the (non-compact) boundary $\partial\Omega$, can one find a harmonic function h_f on Ω which satisfies $h_f(X) \to f(Y)$ as $X \to Y$ for all regular boundary points Y? R. Nevanlinna [Nev] showed in 1925 that this was possible in the case where Ω is a half-plane, and several authors subsequently considered the question further. Results in Chapter 3 will be used to give a complete topological characterization of the sets Ω in which such a Dirichlet problem can always be solved. The second application that we mention here concerns the Radon transform, which itself has applications to tomography. Let f be a real- or complex-valued function on $\mathbf{R}^n$ such that f is integrable on each $(n-1)$-dimensional hyperplane P in $\mathbf{R}^n$. The Radon transform $\widehat{f}$ is defined on the collection $\mathcal{P}^{(n)}$ of all such hyperplanes by $\widehat{f}(P) = \int_P f$ for each P in $\mathcal{P}^{(n)}$, where the integration is with respect to $(n-1)$-dimensional Lebesgue measure on P. An old question concerning the Radon transform was whether there exists a non-constant continuous function f such that $\widehat{f} \equiv 0$ on $\mathcal{P}^{(n)}$. In Chapter 8 we show how Armitage and Goldstein [AG2] used an approximation theorem from Chapter 5 to answer this question: surprisingly, there even exists a non-constant *harmonic* function on $\mathbf{R}^n$ with this property! (When $n = 2$ the question had previously been settled by Zalcman [Zal] using holomorphic approximation.)

These notes began to take shape while I was on sabbatical at McGill University during the calendar year 1992. Some of the early material formed part of an advanced course of lectures entitled *Thin sets and their applications* which I gave there during the Fall Semester. I would like to thank the Department of Mathematics and Statistics at McGill, and especially Prof. K. N. GowriSankaran, for making my visit possible. I am grateful also to Professors David Armitage, Paul Gauthier, Maciej Klimek and Ivan Netuka for their comments on a first draft of these notes. Finally I wish to thank Siobhán Purcell for her assistance in preparing the camera-ready copy and Professor Mícheál Ó Searcóid for the preparation of the diagrams.