0 Review of Thin Sets

The purpose of this preliminary chapter is to give a brief review of facts concerning thin sets which are particularly relevant to harmonic approximation. No proofs will be given, but appropriate references to the books by Helms [Helm] and Doob [Doo] will be supplied.

0.1 Introduction

We use $\overline{A}$, ∂A and A° to denote respectively the closure, boundary and interior of a set A in Euclidean space $\mathbf{R}^n$ $(n \geq 2)$, denote by $|X|$ the Euclidean norm of a point X, and denote by $B(X,r)$ the open ball of centre X and radius r. Also, we define $\phi_n : [0,+\infty) \to \mathbf{R} \cup \{+\infty\}$ by $\phi_2(t) = \log(1/t)$, or $\phi_n(t) = t^{2-n}$ if $n \geq 3$. (We interpret $\phi_n(0)$ as $+\infty$ in either case.) Let Ω be an open set in $\mathbf{R}^n$. A function u on Ω, taking values in $(-\infty,+\infty]$, is called *superharmonic* if:

(i) $u \not\equiv +\infty$ on any component of Ω;
(ii) u is lower semicontinuous, i.e. the set $\{X \in \Omega : u(X) > a\}$ is open for each real number a; and
(iii) $u(X) \geq \mathcal{M}(u;X,r)$ whenever $\overline{B(X,r)} \subseteq \Omega$, where $\mathcal{M}(u;X,r)$ denotes the mean value of u over the sphere $\partial B(X,r)$.

A fundamental example of a superharmonic function on $\mathbf{R}^n$ is given by $\phi_n(|X|)$, which is harmonic on $\mathbf{R}^n \backslash \{O\}$, and which takes the value $+\infty$ at the origin O. It is a consequence of the above definition that a superharmonic function can take the value $+\infty$ only on a rather small set of points. A set A is called *polar* if there is a superharmonic function u on $\mathbf{R}^n$ such that $A \subseteq \{X : u(X) = +\infty\}$. Thus, for example, the set $\{0\}^m \times \mathbf{R}^{n-m}$ is polar if $m \in \{2,3,\ldots,n\}$, as can be seen by considering the superharmonic function

$$X \mapsto \phi_m\left((x_1^2 + x_2^2 + \ldots + x_m^2)^{1/2}\right).$$

However, the hyperplane $\{0\} \times \mathbf{R}^{n-1}$ is not polar. Polar sets always have n-dimensional Lebesgue measure 0.

As the definition suggests, superharmonic functions need not be continuous, even in the extended sense of functions taking values in $[-\infty, +\infty]$. In fact, if $\{Y_k : k \in \mathbf{N}\}$ is a countable dense subset of some ball, where $\mathbf{N} = \{1, 2, \ldots\}$, then a highly discontinuous superharmonic function is defined by

$$u(X) = \sum_{k=1}^{\infty} 2^{-k} \phi_n\left(|X - Y_k|\right) \quad (X \in \mathbf{R}^n).$$

Nevertheless, it is possible to assert that, if u is a superharmonic function on Ω and $Y \in \Omega$, then

$$u(X) \to u(Y) \quad (X \to Y; X \notin E),$$

where the exceptional set E is, in a sense which becomes clear from Wiener's criterion in §0.5, "thin" at Y.

0.2 The Fine Topology

If T_1, T_2 are topologies on the same set and $T_2 \subseteq T_1$, then T_1 is said to be *finer* than T_2, and T_2 is said to be *coarser* than T_1. The *fine topology* of classical potential theory is the coarsest topology on $\mathbf{R}^n$ which makes every superharmonic function on $\mathbf{R}^n$ continuous. It is obtained by taking the intersection of all topologies which make the superharmonic functions continuous. The fine topology clearly contains all open balls, and hence the Euclidean topology. It is strictly finer than the Euclidean topology since, as we have seen, there exist superharmonic functions which are discontinuous with respect to the latter.

A set E is said to be *thin* at a point Y if Y is not a fine limit point of E, i.e. if there is a fine (topology) neighbourhood N of Y such that $E \backslash \{Y\}$ does not intersect N. The classical example of a thin set is the *Lebesgue spine* in $\mathbf{R}^3$ defined by

$$E_c = \{(x, y, z) : x > 0 \text{ and } y^2 + z^2 < \exp(-c/x)\} \quad (c > 0), \tag{0.1}$$

which is thin at O. A polar set is thin at every point of $\mathbf{R}^n$.

Theorem 0.A. *([Helm, 10.3], [Doo, 1.XI.2]) Let Y be a limit point of a set E. Then E is thin at Y if and only if there is a superharmonic function u on an open neighbourhood of Y such that*

$$\liminf_{X \to Y, X \in E} u(X) > u(Y).$$

(Throughout these notes, limit concepts for a function u do not involve the value of u at the point concerned.)

An open set Ω in $\mathbf{R}^n$ will be called *Greenian* if it possesses a Green function $G_\Omega(\cdot,\cdot)$. When $n \geq 3$ all open sets are Greenian. When $n = 2$ an open set Ω is Greenian if and only if its complement is a not a polar set (see [Helm, 8.33]). If u is a non-negative superharmonic function on a Greenian open set Ω such that the greatest harmonic minorant of u on Ω is the zero function, then u is called a *potential* on Ω. In this case there exists a unique (Borel) measure ν_u on Ω such that $u = G_\Omega \nu_u$ where

$$G_\Omega \nu_u(X) = \int_\Omega G_\Omega(Y,X)\, d\nu_u(Y) \quad (X \in \Omega).$$

Any non-negative superharmonic function u on Ω can be written as the sum of its greatest harmonic minorant and a potential of the above form. The measure ν_u is called the *Riesz measure* associated with u. It is given by $\nu_u = -c_n \Delta u$ in the sense of distributions, where $c_n^{-1} = \sigma_n \max\{n-2, 1\}$ and σ_n denotes the surface area of the unit sphere in $\mathbf{R}^n$.

Theorem 0.B. *([Doo, 1.XI.2], cf. [Helm, 10.4]) Let Ω be a Greenian open set and suppose that a set E is thin at a limit point Y in Ω. Then there is a potential u on Ω such that*

$$u(Y) < \liminf_{X \to Y, X \in E} u(X) = +\infty.$$

Corollary 0.C. *([Helm, 10.5]) If a Borel set E is thin at a point Y, then*

$$\frac{\sigma\big(\partial B(Y,r) \cap E\big)}{\sigma\big(\partial B(Y,r)\big)} \to 0 \qquad (r \to 0+),$$

where σ denotes surface area measure on $\partial B(Y,r)$.

Theorem 0.D. *([Helm, 10.14]) If a set E in $\mathbf{R}^2$ is thin at a point Y, then there are arbitrarily small positive values of r such that $\partial B(Y,r) \cap E = \emptyset$.*

Theorem 0.E. *([Helm, 10.9], [Doo, 1.XI.6]) Let $E \subseteq \mathbf{R}^n$. The set of points of E where E is thin forms a polar set.*

We note from Theorem 0.E that, since an $(n-1)$-dimensional hyperplane (or line, if $n = 2$) is non-polar, it cannot be thin at any of its constituent points, in view of its translational symmetries.

0.3 Reduced Functions and Thinness

Let Ω be a Greenian open set, let u be a non-negative superharmonic function on Ω, and let $E \subseteq \Omega$. The *réduite*, or *reduced function*, of u relative to E in Ω is defined by

$$R_u^E(X) = \inf\{v(X) : v \text{ is a superharmonic function on } \Omega,\\ v \geq 0 \text{ on } \Omega, v \geq u \text{ on } E\}$$

when $X \in \Omega$. Its lower regularization, that is

$$\widehat{R}_u^E(X) = \min\left\{\liminf_{Y \to X} R_u^E(Y), R_u^E(X)\right\} \qquad (X \in \Omega),$$

is called the *balayage*, or *regularized reduced function*, of u relative to E in Ω. The balayage is a superharmonic function on Ω. It is obvious that $u \geq R_u^E \geq \widehat{R}_u^E \geq 0$ on Ω, that $u = R_u^E$ on E, and that $u = R_u^E = \widehat{R}_u^E$ on E°. Some further properties of reduced functions are listed below:

(i) R_u^E (and hence also $\widehat{R}_u^E$) is harmonic on $\Omega \backslash \overline{E}$ ([Helm, 7.11]);

(ii) $\widehat{R}_u^E = R_u^E$ on $\Omega \backslash E$ ([Helm, 8.36]);

(iii) $\widehat{R}_u^E = R_u^E$ on Ω if E is open (cf. (ii));

(iv) $\widehat{R}_u^E$ differs from R_u^E at most on a polar set ([Helm, 7.39]);

(v) if F is a polar set, then $\widehat{R}_u^{E \backslash F} = \widehat{R}_u^E$ on Ω ([Helm, 8.37]);

(vi) if (E_k) is an increasing sequence of sets and $E = \cup_k E_k$, then $R_u^{E_k} \uparrow R_u^E$ and $\widehat{R}_u^{E_k} \uparrow \widehat{R}_u^E$ (cf. [Helm, 8.38]).

(All the above properties can also be found in [Doo, 1.VI.3].)

Theorem 0.F. *([Doo, 1.XI.10]) Let Ω be a Greenian open set. Then there is a bounded continuous potential $u^\#$ on Ω with the property that a set E is thin at a point Y in Ω if and only if $\widehat{R}_{u^\#}^E(Y) < u^\#(Y)$.*

0.4 Thin Sets and the Dirichlet Problem

We refer to [Helm, Chapters 8,9] and [Doo, 1.VIII] for accounts of the Perron-Wiener-Brelot solution to the Dirichlet problem on a Greenian open set Ω with boundary function $f : \partial^*\Omega \to [-\infty, +\infty]$. Here $\partial^*\Omega$ denotes $\partial\Omega$ if Ω is bounded, or $\partial\Omega \cup \{\infty\}$ if Ω is unbounded, and ∞ denotes the Alexandroff point for $\mathbf{R}^n$. We recall that a function u on Ω is said to be in the *upper* (resp. *lower*) *PWB class* if, on each component of Ω, either $u \equiv +\infty$ (resp. $u \equiv -\infty$) or u is superharmonic (resp. subharmonic) and bounded below (resp. above), and if

$$\liminf_{X \to Y} u(X) \geq f(Y) \qquad \left(\text{resp.} \limsup_{X \to Y} u(X) \leq f(Y)\right) \qquad (Y \in \partial^*\Omega).$$

Further, the infimum (resp. supremum) of the upper (resp. lower) PWB class is denoted by $\overline{H}_f^\Omega$ (resp. $\underline{H}_f^\Omega$). If $\overline{H}_f^\Omega$ and $\underline{H}_f^\Omega$ are identical and harmonic on Ω, then we denote them by H_f^Ω. In this case f is said to be *resolutive* for Ω, and H_f^Ω is called the *PWB solution* for f. A point Y in $\partial^*\Omega$ is called *regular* if $H_f^\Omega(X) \to f(X)$ as $X \to Y$ for every continuous function $f : \partial^*\Omega \to \mathbf{R}$. Otherwise Y is called *irregular*. The set Ω is called *regular* if every point in $\partial^*\Omega$ is regular. For each X in Ω there is a unique (Borel) measure $\mu_{\Omega,X}$ on $\partial^*\Omega$ such that

$$H_f^\Omega(X) = \int_{\partial^*\Omega} f(Y)\, d\mu_{\Omega,X}(Y) \qquad (X \in \Omega)$$

for every resolutive boundary function f. The measure $\mu_{\Omega,X}$ is called *harmonic measure* for Ω and X. If Ω is connected, then the class of Borel subsets of $\partial^*\Omega$ which have zero $\mu_{\Omega,X}$-measure is independent of X. In connection with the following results we emphasize that $\partial\Omega$ denotes the *Euclidean* boundary of Ω, and so does not include ∞ even if Ω is unbounded.

Theorem 0.G. *([Helm, 10.12], [Doo, 1.XI.12]) Let Ω be a Greenian open set and let $Y \in \partial\Omega$. Then Y is a regular boundary point for the Dirichlet problem on Ω if and only if $\mathbf{R}^n \setminus \Omega$ is not thin at Y.*

Theorem 0.H. *([Doo, 1.XI.13]) Let Ω be a connected Greenian open set. Then the set of points of $\partial\Omega$ at which Ω is thin forms a set of zero harmonic measure for Ω.*

We also record here the relationship between reduced functions and Dirichlet solutions.

Theorem 0.I. *([Helm, 9.25], [Doo,1.VIII.10]) Let Ω be a Greenian open set, let ω be an open subset of Ω, and let u be a positive superharmonic function on Ω. Then $R_u^{\Omega \setminus \omega} = H_{u'}^\omega$ on ω, where*

$$u'(X) = \begin{cases} u(X) & (X \in \Omega \cap \partial\omega) \\ 0 & (X \in \partial\Omega \cap \partial\omega; X = \infty \text{ if } \omega \text{ is unbounded}). \end{cases}$$

0.5 Wiener's Criterion

Let Ω be an open set in $\mathbf{R}^n$ with Green function $G_\Omega(\cdot,\cdot)$, let $E \subseteq \mathbf{R}^n$ and $Y \in \Omega$, and let α denote a fixed number satisfying $\alpha > 1$. For each positive integer k we define

$$E_k = \{X \in E : \alpha^k \le \phi_n(|X-Y|) \le \alpha^{k+1}\}.$$

Also, we choose k' such that the closed ball $\{X : \alpha^{k'} \le \phi_n(|X-Y|)\}$ is contained in Ω. In what follows, $\mathcal{C}^*$ denotes outer capacity with respect to Ω (see [Helm, Chap. 7] or [Doo, 1.XIII]). When $n \ge 3$ we may take $\Omega = \mathbf{R}^n$, in which case $\mathcal{C}^*$ is outer Newtonian capacity.

Theorem 0.J. *([Helm, 10.21], [Doo, 1.XI.3 and 1.XIII.17]) The following are equivalent:*

(i) E is thin at Y;

(ii) $\sum_{k=k'}^{\infty} \alpha^k \mathcal{C}^*(E_k) < +\infty$ (Wiener's criterion);

(iii) $\int_{\alpha^{k'}}^{\infty} \mathcal{C}^*(\{X \in E : \phi_n(|X-Y|) > t\})\,dt < +\infty$;

(iv) $\widehat{R}^E_{G_\Omega(Y,\cdot)} \ne G_\Omega(Y,\cdot)$ (unequal as functions).

Theorem 0.K. *([Doo, 1.XI.4]) Let Ω be a Greenian open set, and let u be a positive superharmonic function on Ω with associated Riesz measure ν_u. Then $u/G_\Omega(Y,\cdot)$ has fine limit $\nu_u(\{Y\})$ at Y.*

1 Approximation on Compact Sets

1.1 Introduction

If Ω is an open set in $\mathbf{C}$ or $\mathbf{R}^n (n \geq 2)$, then we will use Ω^* to denote the Alexandroff, or one-point, compactification of Ω, and will use $\mathcal{A}$ to denote the ideal point. Thus $\Omega^* = \Omega \cup \{\mathcal{A}\}$, and a set A is open in Ω^* if either A is an open subset of Ω or $A = \Omega^* \backslash K$, where K is a compact subset of Ω. In the special case where Ω is $\mathbf{C}$ or $\mathbf{R}^n$ we continue to write ∞ for $\mathcal{A}$.

If A is a subset of $\mathbf{C}$, we denote by $\mathrm{Hol}(A)$ the collection of all functions which are holomorphic on an open set containing A. Historically the following result (essentially in [Run]; cf. [Con, pp.198, 201]) can be regarded as the starting point of the theory of holomorphic approximation.

Runge's Theorem (1885). *Let Ω be an open subset of $\mathbf{C}$ and K be a compact subset of Ω. The following are equivalent:*
(a) for each f in $\mathrm{Hol}(K)$ and each positive number ϵ, there exists g in $\mathrm{Hol}(\Omega)$ such that $|g - f| < \epsilon$ on K;
(b) $\Omega^* \backslash K$ is connected.

Condition (b) above is equivalent to asserting that no component of $\Omega \backslash K$ is relatively compact in Ω. Also, when $\Omega = \mathbf{C}$, this condition is clearly equivalent to saying that $\mathbf{C} \backslash K$ is connected.

We record below one further important development in the theory of holomorphic approximation, which deals with approximation of a much larger class of functions on a given compact set K. It can be found in [Mer] or [Rud, Chap. 20]. As usual we denote by $C(A)$ the collection of all complex- (or real-, depending on the context) valued continuous functions on a set A.

Mergelyan's Theorem (1952). *Let K be a compact set in $\mathbf{C}$. The following are equivalent:*
(a) for each f in $C(K) \cap \mathrm{Hol}(K^o)$ and each positive number ϵ, there is an

entire function (and hence, by suitably truncating the Taylor series, a polynomial) g such that $|g - f| < \epsilon$ on K;
(b) $\mathbf{C}\backslash K$ is connected.

Turning now to the history of harmonic approximation in $\mathbf{R}^n (n \geq 2)$, we take as our starting point the following result [Wal, p.541].

Walsh's Theorem (1929). *Let K be a compact set in $\mathbf{R}^n$ such that $\mathbf{R}^n\backslash K$ is connected. Then, for each function u which is harmonic on an open set containing K and each positive number ϵ, there is a harmonic polynomial v such that $|v - u| < \epsilon$ on K.*

Important progress concerning uniform harmonic approximation was made in the 1940's, in which connection we mention particularly the contributions by Keldyš [Kel], Landkof (see the references in [Lan]), Brelot [Bre1], and Deny [Den1], [Den2]. However, somewhat surprisingly, the analogue of Runge's Theorem (as stated above) for harmonic approximation in $\mathbf{R}^n$ was obtained rather more recently. In this chapter we present analogues of both Runge's Theorem and Mergelyan's Theorem for harmonic functions. First, however, we deal with the question of local harmonic approximation, i.e. approximation by harmonic functions defined merely on some neighbourhood of a given compact set.

1.2 Local Approximation on Compact Sets with Empty Interior

In this section and the next we will be concerned with uniform approximation of functions on a compact set K in $\mathbf{R}^n$ by functions harmonic on a neighbourhood of K. Clearly the functions to be approximated must be continuous on K and harmonic on the interior K°. It will be convenient to denote by $\mathcal{H}(A)$ the collection of all functions which are harmonic on some open set containing A. The question we are concerned with is this. Which compact sets K have the property that every u in $C(K) \cap \mathcal{H}(K^\circ)$ can be uniformly approximated by functions in $\mathcal{H}(K)$? This question simplifies if we restrict our attention to compact sets K with empty interior, so this special case will be treated first. In §1.3 the more general question will be dealt with. Of course, if $K^\circ = \emptyset$, then we are approximating arbitrary continuous functions on K.

Theorem 1.1. *Let K be a compact subset of $\mathbf{R}^n$ such that $K^\circ = \emptyset$. The following are equivalent:*

(a) for each f in $C(K)$ and each positive number ϵ, there exists h in $\mathcal{H}(K)$ such that $|h - f| < \epsilon$ on K;
(b) $\mathbf{R}^n \backslash K$ is nowhere thin.

Before proving this theorem we present an example of a compact set K which has empty interior yet fails to satisfy condition (b).

Example 1.2. Let $\{Y_k : k \in \mathbf{N}\}$ be a dense subset of $[0,1]^{n-1} \times (0,1]$, and define

$$u(X) = \sum_{k=1}^{\infty} 2^{-k} \phi_n(|X - Y_k|) \qquad (X \in \mathbf{R}^n),$$

$$E = \left([0,1]^{n-1} \times \{0\}\right) \cup \left\{(X', x_n) \in [0,1]^{n-1} \times (0,1] : u(X', x_n) \le \phi_n(x_n)\right\}$$

and

$$K = \left\{(x_1, \ldots, x_n) \in \mathbf{R}^n : (x_1, \ldots, x_{n-1}, |x_n|) \in E\right\}.$$

Then u is a superharmonic function on $\mathbf{R}^n$, and the lower semicontinuity of u ensures that E is closed. Thus K is compact and, because $u(Y_k) = +\infty$ for each k, the interior K° is empty. (The set K is an example of what is sometimes called a "Swiss cheese": cf. [Rot1, p.96].) If $Z \in (0,1)^{n-1} \times \{0\}$ and $X = (X', x_n)$, where $x_n > 0$, then

$$u(X) > \phi_n(x_n) \ge \phi_n(|X - Z|) \qquad (X \in (0,1)^n \backslash E).$$

Since the Riesz measure associated with u does not charge $\{Z\}$, it follows easily from Theorem 0.K that $(0,1)^n \backslash E$ is thin at Z, and hence $\mathbf{R}^n \backslash K$ is thin at Z.

In both parts of the proof below, K will denote a compact set with empty interior, B will be a fixed open ball which contains K, and reductions will be with respect to superharmonic functions on B. Also, we define

$$U_m = \{X \in \mathbf{R}^n : \operatorname{dist}(X, K) < 1/m\} \qquad (m \in \mathbf{N}).$$

Proof that (b) implies (a). Suppose that $\mathbf{R}^n \backslash K$ is nowhere thin, let $f \in C(K)$ and $\epsilon > 0$. There exist (see [Helm, 8.10]) positive continuous superharmonic functions u_1, u_2 on B such that

$$\left|f - (u_1 - u_2)\right| < \epsilon/2 \text{ on } K. \tag{1.1}$$

We know that

$$R_{u_k}^{B \backslash U_m}(X) \uparrow R_{u_k}^{B \backslash K}(X) \qquad (X \in B; k = 1, 2; m \to \infty). \tag{1.2}$$

Now let v_k be a positive superharmonic function on B such that $v_k \geq u_k$ on $B\backslash K$, where $k \in \{1,2\}$. It follows by fine continuity that $v_k - u_k \geq 0$ on K, since $B\backslash K$ is non-thin at each X in K and so every fine neighbourhood of such a point X meets $B\backslash K$. Thus $v_k \geq u_k$ on B and hence

$$u_k = R_{u_k}^{B\backslash K} \qquad \text{on } B.$$

From (1.2) we see that

$$R_{u_k}^{B\backslash U_m}(X) \uparrow u_k(X) \qquad (X \in K; k = 1,2; m \to \infty)$$

and, since u_k is continuous and K is compact, Dini's theorem shows that this convergence is uniform on K. Thus there exists m' such that

$$u_k(X) \geq R_{u_k}^{B\backslash U_{m'}}(X) > u_k(X) - \epsilon/2 \qquad (X \in K; k = 1,2). \tag{1.3}$$

It follows from (1.1) and (1.3) that

$$f - R_{u_1}^{B\backslash U_{m'}} + R_{u_2}^{B\backslash U_{m'}} < f - u_1 + \epsilon/2 + u_2 < \epsilon \qquad \text{on } K,$$

and similarly

$$f - R_{u_1}^{B\backslash U_{m'}} + R_{u_2}^{B\backslash U_{m'}} > -\epsilon \qquad \text{on } K.$$

Since the above reduced functions are harmonic on the open set $U_{m'}$ which contains K, the argument is complete.

Proof that (a) implies (b). Suppose that condition (a) of Theorem 1.1 holds, let $u^\#$ be the bounded continuous potential on B described in Theorem 0.F, and let $\epsilon > 0$. By hypothesis there exists h_ϵ in $\mathcal{H}(K)$ such that

$$|h_\epsilon - u^\#| < \epsilon \qquad \text{on } K. \tag{1.4}$$

By continuity the above inequality remains true on the open set U_m for all sufficiently large m. Thus, solving the Dirichlet problem in U_m, we obtain

$$-\epsilon \leq h_\epsilon - H_{u^\#}^{U_m} = h_\epsilon - R_{u^\#}^{B\backslash U_m} \leq \epsilon \qquad \text{on } K$$

(See Theorem 0.I) Letting $m \to \infty$, it follows that

$$\left| h_\epsilon - R_{u^\#}^{B\backslash K} \right| \leq \epsilon \qquad \text{on } K.$$

Combining this with (1.4) we see that

$$\left| u^\# - \widehat{R}_{u^\#}^{B\backslash K} \right| = \left| u^\# - R_{u^\#}^{B\backslash K} \right| < 2\epsilon \qquad \text{on } K.$$

Since ϵ can be arbitrarily small,

$$u^\# = \widehat{R}_{u^\#}^{B\backslash K} \qquad \text{on } K.$$

From Theorem 0.F we can conclude that $B\backslash K$, and hence $\mathbf{R}^n\backslash K$, is not thin at any point of K. Certainly $\mathbf{R}^n\backslash K$ is not thin at any point of $\mathbf{R}^n\backslash K$, so condition (b) of the theorem is established.