Proper affine isometric actions of amenable groups

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A property of a (countable) group Γ of relevance both in harmonic analysis and operator algebras is the so-called *Haagerup's approximation property* (see [Cho], [JV], [Ro]): one possible definition is to say that the abelian C^*-algebra $c_0(\Gamma)$ has an approximate unit consisting of positive definite functions on Γ.

On the other hand, in §§7.A and 7.E of his book [Gr], M. Gromov introduced the following definition (and dubious pun): Γ is *a-T-menable* if Γ admits a proper affine isometric action on some Hilbert space $\mathcal{H}$, where "proper action" means that, for any bounded subsets B, C in $\mathcal{H}$, the set of elements $g \in \Gamma$ such that $\alpha(g)B$ meets C is finite. (This non-standard sense of properness is relevant for actions on general metric spaces.)

Our first result is that the two concepts are actually equivalent.

Lemma. *For a group Γ, the following are equivalent:*

(i) Γ has the Haagerup approximation property;

(ii) Γ admits a proper function of conditionally negative type;

(iii) Γ is a-T-menable.

Proof. (i) $\Leftrightarrow$ (ii) is due to Akemann and Walter (Theorem 10 in [AW]).

(ii) $\Rightarrow$ (iii). Let ψ be a proper function of conditionally negative type on Γ. By Proposition 14 of [HV], there exists an affine isometric action α of Γ on a Hilbert space $\mathcal{H}$ such that, for any $g \in \Gamma$,

$$\psi(g) = \|\alpha(g)(0)\|^2.$$

We claim that α is a proper action. To see this, it is enough to check that, for any $R > 0$, the set $F_R = \{g \in \Gamma : \alpha(g)B_R \cap B_R \neq \emptyset\}$ is finite (here B_R denotes the closed ball with radius R centered at 0).

For $g \in G$, denote by $\pi(g)$ the linear part of $\alpha(g)$, so that $\alpha(g)(0)$ is its translation part, i.e.,

$$\alpha(g)\xi = \pi(g)\xi + \alpha(g)(0)$$

for any $\xi \in \mathcal{H}$.

If $g \in F_R$, we find $\xi \in B_R$ such that $\|\alpha(g)(\xi)\| \leq R$, which implies $\|\alpha(g)(0)\| \leq R + \|\pi(g)(\xi)\| \leq 2R$ or $\psi(g) \leq 4R^2$. Thus F_R is contained in $\{g \in \Gamma : \psi(g) \leq 4R^2\}$, a finite set by assumption.

(iii) $\Rightarrow$ (ii). If α is a proper isometric action of Γ on $\mathcal{H}$, then the function $\psi : \Gamma \to \mathbb{R}; g \mapsto \|\alpha(g)(0)\|^2$ is of conditionally negative type (see n^o 13 of Chapter 5 in [HV]). On the other hand, it is clear that ψ is proper. □

During the problem session at the Oberwolfach Conference on "Novikov conjectures, index theorems and rigidity,"[1] (Sept. 5–11, 1993), Gromov asked whether any amenable group is a-T-menable; doing so, he advertised a Question in §7.E of [Gr]. Our main result is that Gromov's question has an affirmative answer.

Let us fix more notation. We denote by λ_Γ the left regular representation of Γ on $\ell^2(\Gamma)$, and by π_Γ the direct sum of countably many copies of λ_Γ, acting on

$$\mathcal{H}_\Gamma = \ell^2(\Gamma) \oplus \ell^2(\Gamma) \oplus \ell^2(\Gamma) \oplus \cdots$$

(countably many summands).

Proposition. *Let Γ be a countable amenable group. Then Γ admits a proper affine isometric action α on $\mathcal{H}_\Gamma$, such that the linear part of α is π_Γ.*

Proof. Let $(F_k)_{k\geq 1}$ be an increasing family of finite subsets of Γ, such that $\bigcup_{k=1}^{\infty} F_k = \Gamma$. By Følner's property, we find for any $k \geq 1$ a finite subset U_k of Γ, such that for any $g \in F_k$,

$$\frac{|gU_k \,\triangle\, U_k|}{|U_k|} < 2^{-k}.$$

Let ξ_k be the normalized characteristic function of U_k, i.e.,

$$\xi_k(x) = \begin{cases} |U_k|^{-\frac{1}{2}}, & \text{if } x \in U_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\|\lambda_\Gamma(g)\xi_k - \xi_k\|^2 = \frac{|gU_k \,\triangle\, U_k|}{|U_k|}$$

for any $g \in \Gamma$.

[1] The third author thanks the organizers for inviting him to that stimulating week!

For $g \in \Gamma$, set now $b(g) = \bigoplus_{k=0}^{\infty} k(\lambda_\Gamma(g)\xi_k - \xi_k)$. This series converges in $\mathcal{H}_\Gamma$ because, for $g \in F_n$, we have

$$\left\| \bigoplus_{k=n}^{\infty} k(\lambda_\Gamma(g)\xi_k - \xi_k) \right\|^2 = \sum_{k \geq n} k^2 \left\|\lambda_\Gamma(g)\xi_k - \xi_k\right\|^2 \leq \sum_{k \geq n} k^2 2^{-k} < \infty.$$

It is immediate to check that b is a 1-cocycle with respect to π_Γ, i.e., for any $g, h \in \Gamma$, $b(gh) = \pi_\Gamma(g)(b(h)) + b(g)$. Thus, defining $\alpha(g) : \mathcal{H}_\Gamma \to \mathcal{H}_\Gamma$ by $\alpha(g)\xi = \pi_\Gamma(g)\xi + b(g)$, we see that α is an affine isometric action of Γ on $\mathcal{H}_\Gamma$, with linear part π_Γ.

Now, we define an function ψ conditionally of negative type on Γ by $\psi(g) = \|b(g)\|^2$, and we claim that ψ is a proper function. This amounts to proving that, for $R \geq 0$, the set $C_R = \{g \in \Gamma : \|b(g)\| \leq R\}$ is finite. To see this, we fix $N \in \mathbb{N}$ such that $R \leq N$. Then, for $g \in C_R$:

$$N^2 \left\|\lambda_\Gamma(g)\xi_N - \xi_N\right\|^2 \leq \|b(g)\|^2 \leq R^2,$$

hence $|gU_N \triangle U_N| \leq |U_N|$ or $\frac{|U_N|}{2} \leq |U_N \cap gU_N|$. But the set of h's in Γ such that $\frac{|U_N|}{2} \leq |U_N \cap hU_N|$ is clearly finite.

From the fact that ψ is a proper function conditionally of negative type, one deduces the fact that α is a proper action as in the implication (ii) $\Rightarrow$ (iii) of the Lemma. $\square$

Remark. It is known (see e.g. [HV, Chapter 4]) that a countable group Γ does *not* have Kazhdan's property (T) if and only if Γ admits an affine isometric action with *unbounded* orbits on some Hilbert space. For Γ a countably infinite amenable group, the action α just constructed appeared in [Che] to give an explicit example of such an affine isometric action with unbounded orbits. Guichardet also proved that any such group admits an affine isometric action with unbounded orbits on $\ell^2(G)$, the linear part of which is λ_Γ (see [Gu], Cor. 2.4 in Chapter III; the proof, appealing to the closed graph theorem, is not constructive, so we do not know whether such an action is proper or not).

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Bounded K–theory and the Assembly Map in Algebraic K–theory

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Introduction

Since their introduction 20 years ago, the evaluation of Quillen's higher algebraic K–groups of rings [35] has remained a difficult problem in homotopy theory. One does have Quillen's explicit evaluation of the algebraic K–theory of finite fields [36] and Suslin's theorem about the K–theory of algebraically closed fields [41]. In addition, Quillen's original paper [35] gave a number of useful formal properties of the algebraic K–groups: localization sequences, homotopy property for the K–theory of polynomial rings, reduction by resolution, and reduction by "devissage". These tools provide, for a large class of commutative rings, a fairly effective procedure which reduces the description of the K–theory of the commutative ring to that of fields. The K–theory of a general field remains an intractable problem due to the lack of a good Galois descent spectral sequence, although by work of Thomason [45] one can understand its so–called Bott–periodic localization. In the case of non–commutative rings, the formal properties of Quillen are not nearly as successful as in the commutative case.

A central theme in the subject has been the relationship with properties of manifolds which are not homotopy invariant; here the ring in question is usually the group ring $\mathbb{Z}[\Gamma]$, with Γ the fundamental group of a manifold. Examples include Wall's finiteness obstruction [49], the s–cobordism theorem of Barden–Mazur–Stallings [31], Hatcher–Wagoner's work [24] on the connection between K_2 and the homotopy type of the pseudoisotopy space, and finally Waldhausen's description [48] of the pseudoisotopy space in terms of the K–theory of "rings up to homotopy". Since the rings $\mathbb{Z}[\Gamma]$ are typically not commutative, the reduction methods of [35] are not adequate for the description of the K–theory of $\mathbb{Z}[\Gamma]$.

Fortunately, in the case of group rings, there is a reasonable conjecture concerning the structure of the K–theory of $\mathbb{Z}[\Gamma]$, or more generally $A[\Gamma]$, where A is a commutative ring. Let Γ be any group, let $B\Gamma_+$ denote

[1] Supported in part by NSF DMS 8907771.A01

its classifying space with a disjoint basepoint added, and let $\underset{\sim}{K}A$ denote the connective spectrum associated to the symmetric monoidal category of finitely generated projective A–modules. (We will allow ourselves to use the language of spectra freely, in this introduction and throughout the paper; see [1] or [30] for a discussion.) The homotopy groups of $\underset{\sim}{K}A$ are precisely the higher algebraic K–groups of Quillen. Then we have the "assembly map"

$$\alpha \ : \ B\Gamma_+ \wedge \underset{\sim}{K}A \longrightarrow KA[\Gamma]$$

(see [27] or [47]), which is constructed out of the group homomorphism

$$\Gamma \times GL_n A \to GL_1 A[\Gamma] \times GL_n A \xrightarrow{\otimes} GL_n A[\Gamma] \ .$$

Here, Γ is included in $GL_1 A[\Gamma]$ by recognizing that any $\gamma \in \Gamma$ can be viewed as a unit in the ring $A[\Gamma]$, and $\otimes$ is the homomorphism

$$GL_1(A[\Gamma]) \times GL_n(A[\Gamma] \underset{A}{\otimes} A) \to GL_n(A[\Gamma]) \ ; \ (M,N) \to M \otimes N \ .$$

A preliminary conjecture about $\underset{\sim}{K}A[\Gamma]$ is that α is an equivalence. This would provide a complete description of $\underset{\sim}{K}A[\Gamma]$ in terms of its constituent parts $\underset{\sim}{K}A$ and $B\Gamma_+$, and would in particular yield a spectral sequence with $E^2_{p,q} = H_p(\Gamma, K_q A)$ converging to $K_{p+q}A[\Gamma]$. The conjecture is however known to fail in general. If Γ is finite, then even with the coefficient ring $\mathbb{C}$, one can show by direct calculation that the map fails to be an isomorphism. On the other hand, if A contains nilpotent elements, one can show that even for the group $\Gamma = \mathbb{Z}$ (the conjecture holds for A regular and $\Gamma = \mathbb{Z}$ by the methods of Quillen [35]), α fails to be an equivalence due to the presence of Nil–groups (see [5]), in the K–theory of the polynomial ring over A. However, there are no known counter–examples to the conjecture with A regular and Γ a group which admits a finite classifying space. In [47], Waldhausen proves that α is an equivalence, when A is regular, for a large class of groups built from $\mathbb{Z}$ by processes of amalgamated product and extension by $\mathbb{Z}$. The rationalized form of the conjecture ($\pi_i(\alpha) \underset{\mathbb{Z}}{\otimes} \mathrm{id}_{\mathbb{Q}}$ is an isomorphism) is much more approachable, and has been studied by many people. Quinn for instance, has shown that this rationalized form holds when Γ is a torsion free Bieberbach group [37], and Bökstedt–Hsiang–Madsen [6] have shown that the $\pi_i(\alpha) \underset{\mathbb{Z}}{\otimes} \mathrm{id}_{\mathbb{Q}}$ is injective for any group with finitely generated homology. Low dimensional integral information has also been obtained by Farrell–Hsiang [16,17]. More recently, the startling results of Farrell–Jones [18,19] have given a complete description of the pseudo–isotopy space of a closed compact manifold admitting a Riemannian metric with negative curvature; one consequence of this is that $\pi_i(\alpha) \underset{\mathbb{Z}}{\otimes} \mathrm{id}_{\mathbb{Q}}$ is an isomorphism for the fundamental groups of these manifolds. The method here relies on the properties of geodesic flow on the tangent bundle to the manifold. Their method will generalize to a larger class of groups, including fundamental groups of manifolds admitting a Riemannian metric with non–positive curvature and cocompact discrete torsion-free subgroups of Lie groups, but always relies on fairly precise control on the geometry (i.e., metric structure) of the manifold.

In this paper, we will study the map α directly, without rationalization. We describe our method. Suppose G is a finite group. (Note that our results will certainly not apply to finite groups; it is included here for motivational purposes.) Then the group ring $A[G]$ can be viewed as the algebra of G–invariant A–linear transformations from $A[G]$ to $A[G]$, where G acts on $\mathrm{Hom}_A(A[G], A[G])$ by the conjugation action $(gf)(x) = gf(g^{-1}x)$. Thus, if $|G| = n$, $A[G]$ is the invariant subring of a group action of G on the $n \times n$ matrices over A, given by conjugating by a subgroup of the permutation matrices in $GL_n(A)$. From this fact, one derives that $\underset{\sim}{K}A[G]$ can be obtained as the fixed point spectrum $(\underset{\sim}{K}A)^G$ of an action of G on $\underset{\sim}{K}A$. We now recall the notion of the *homotopy fixed point set* of an action of a group G on a space (or spectrum) X. For any group G, let EG denote a contractible space on which G acts freely. Then we can equip the function space (or spectrum) $F(EG, X)$ with the conjugation G–action $(gf)(z) = gf(g^{-1}z)$. There is of course the equivariant map $EG \to$ point, which induces an equivariant map

$$\varepsilon : X \to F(\text{point}, X) \to F(EG, X) \ ,$$

and because of the contractibility of EG, this map is a homotopy equivalence, although not necessarily a G–homotopy equivalence. We denote $F(EG, X)^G$ by X^{hG}, and refer to it as the homotopy fixed point set of X; ε^G is now a map from X^G to X^{hG}. The advantage of X^{hG} over X^G is that X^{hG} is computable from BG and X in an explicit way; for instance, if G acts trivially on X, $X^{hG} = F(BG_+, X)$. More generally, there is a spectral sequence with $E_2^{p,q} = H^{-p}(G, \pi_q X)$ converging to $\pi_{p+q}(X^{hG})$. We informally say that X^{hG} can be constructed in a "homotopy theoretic" way out of the constituent pieces BG_+ and X, as $BG_+ \wedge \underset{\sim}{K}A$ is built in a homotopy–theoretic way out of BG_+ and $\underset{\sim}{K}A$. We now obtain a map

$$\underset{\sim}{K}A[G] = (\underset{\sim}{K}A)^G \to (\underset{\sim}{K}A)^{hG} \ ,$$

which we use as a detecting device for the assembly map α. Atiyah [2] has shown that it is an effective such device when $A = \mathbb{C}$. In our work, however, we will be exclusively interested in groups Γ which admit a finite classifying space; in particular, they are torsion–free. In the case we can still describe

$A[\Gamma]$ as $\mathrm{Hom}_A(A[\Gamma], A[\Gamma])^\Gamma$, i.e., the fixed point set of an action of Γ on a ring of infinite matrices, which we write $M_\infty(A) = \mathrm{Hom}_A(A[\Gamma], A[\Gamma])$. Unfortunately, the K–theory spectrum of $M_\infty = \mathrm{Hom}_A(A[\Gamma], A[\Gamma])$ can easily be shown to be contractible; elementary properties of the $(-)^{h\Gamma}$ construction (see, e.g., [12]) show that $(\underset{\sim}{K} M_\infty A)^{h\Gamma}$ is therefore contractible. To remedy this, one might attempt to replace $M_\infty(A)$ by a Γ–invariant subring containing $A[\Gamma]$, and whose K–theory spectrum allows one to detect more. Roughly speaking, this is the procedure we use in this paper.

Precisely what we do is sketched as follows. We recall that E. Pedersen and C. Weibel [33], [34], have introduced, for a ring R and metric space X, the *bounded K–theory* $\underset{\sim}{K}(X, R)$; $\underset{\sim}{K}(X, R)$ is a spectrum, has appropriate covariant functoriality properties in both X and R, and when applied to the Euclidean space E^k produces a k–fold developing of the K–theory spectrum of R, equivalent to the Gersten–Wagoner k–fold delooping [22], [46]. When a group Γ acts on a metric space, we introduce a related *equivariant bounded K*–theory spectrum, $\underset{\sim}{K}^\Gamma(X; R)$; it is a spectrum with Γ–action. Viewed as a spectrum without Γ–action, $\underset{\sim}{K}^\Gamma(X; R)$ is equivalent to the original Pedersen–Weibel construction $\underset{\sim}{K}(X; R)$. In general, its fixed point set is difficult to describe. However, suppose X is a compact Riemannian manifold; with $\pi_1(X) = \Gamma$. Then the universal cover of X becomes a Riemannian manifold with free, isometric Γ–action, and the Γ–fixed point set of $\underset{\sim}{K}^\Gamma(X, R)$ is equivalent to the K–theory spectrum $\underset{\sim}{K}(R[\Gamma])$. We have thus achieved the construction of a spectrum with Γ–action with $\underset{\sim}{K}(R[\Gamma])$ as fixed point set, and hence obtain a map

$$\varepsilon^\Gamma : \underset{\sim}{K}(R[\Gamma]) \to \underset{\sim}{K}^\Gamma(X, R)^{h\Gamma} .$$

An elementary (but not as convenient for our purpose as the one given in the paper) Γ–homotopy equivalent version of the $\underset{\sim}{K}^\Gamma(X; R)$–construction can be described as follows. Let Γ be a finitely generated group, and Ω a finite generating set for Γ. For any $\gamma \in \Gamma$, let

$$\ell(\gamma) = \min\{n \,|\, \text{there is a word } w_1^{\pm 1} \dots w_n^{\pm 1} \text{ which is equal to } \alpha\}.$$

For any $\gamma \in \Gamma$ and $\ell \geq 0$, set $N_\ell(\gamma) = \mathrm{span}\{\bar{\gamma} \,|\, \ell(\gamma^{-1}\bar{\gamma}) \leq \ell\}$. Define $M^b(A) \subseteq \mathrm{Hom}\,(A[\Gamma], A[\Gamma])$ to be the subgroup of all $f : A[\Gamma] \to A[\Gamma]$ so that there exists some ℓ so that $f(\gamma) \in N_\ell(\gamma)$ for all $\gamma \in \Gamma$; $M^b(A)$ is of course a subring, closed under the Γ–action and it contains $A[\Gamma]$. One can show that when $\Gamma = \pi_1(X)$ as above, $\underset{\sim}{K}^\Gamma(X, R)$ is equivariantly equivalent to $\underset{\sim}{K} M^b(A)$ with the Γ action induced from the one on $M^b(A)$.

Having obtained our detecting map ε^Γ, we must find some way to evaluate it on the assembly map. We choose to do this by realizing the assembly map as the induced map on fixed point sets of an equivariant map with target $\underset{\sim}{K}^\Gamma(X;R)$. We define, for any spectrum A, a functor ${}^b\underset{\sim}{h}^{\ell f}(-,A)$ defined on an appropriate category of metric spaces, and a natural transformation ${}^b\underset{\sim}{h}^{\ell f}(-,\underset{\sim}{K}(R)) \to \underset{\sim}{K}(-,R)$. When $\underset{\sim}{A}$ is an Eilenberg–MacLane spectrum ${}^b\underset{\sim}{h}^{\ell f}(-,A)$ is closely related to Borel–Moore homology [7]. When X is equipped with an isometric Γ–action ${}^b\underset{\sim}{h}^{\ell f}(-,\underset{\sim}{K}(R))$ becomes a spectrum with Γ–action, and we obtain an equivariant natural transformation ${}^b\underset{\sim}{h}^{\ell f}(-,\underset{\sim}{K}(R)) \to \underset{\sim}{K}^\Gamma(-,R)$. The functor ${}^b\underset{\sim}{h}^{\ell f}(-,\underset{\sim}{K}(R))$ on a large (including all cases of interest to us) category of metric spaces with Γ–action has two crucial properties. The first is that when the Γ–action is free and properly discontinuous, ${}^b\underset{\sim}{h}^{\ell f}(X,\underset{\sim}{K}(R))^\Gamma \cong {}^b\underset{\sim}{h}^{\ell f}(\Gamma\backslash X,\underset{\sim}{K}(R))$. In particular, when $\Gamma\backslash X$ is compact, ${}^b\underset{\sim}{h}^{\ell f}(\Gamma\backslash X,\underset{\sim}{K}(R))$ reduces to ordinary homology of $\Gamma\backslash X$ with coefficients in $\underset{\sim}{K}(R)$, i.e., $(\Gamma\backslash X)_+ \wedge \underset{\sim}{K}(R)$. The second crucial property is that the natural map $\varepsilon^\Gamma : {}^b\underset{\sim}{h}^{\ell f}(X,\underset{\sim}{K}(R))^\Gamma \to {}^b\underset{\sim}{h}^{\ell f}(X,\underset{\sim}{K}(R))^{h\Gamma}$ is an equivalence. Now suppose that X is a compact Riemannian manifold, $\pi_1(X) = \Gamma$, and that $\tilde{X}$ is contractible. Then the natural transformation ${}^b\underset{\sim}{h}^{\ell f}(-,\underset{\sim}{K}(R)) \to \underset{\sim}{K}^\Gamma(-,R)$, by restricting to fixed point sets, gives rise to a map $X_+ \wedge \underset{\sim}{K}(R) \to \underset{\sim}{K}(R[\Gamma])$. Since $\tilde{X}$ is contractible, X is a model for $B\Gamma$, and this map can be identified with the assembly map. We now have the commutative diagram

$$\begin{array}{ccc}
B\Gamma_+ \wedge \underset{\sim}{K}(R) & \longrightarrow & \underset{\sim}{K}(R[\Gamma]) \\
\wr\downarrow & & \downarrow \\
{}^b\underset{\sim}{h}^{\ell f}(\tilde{X},\underset{\sim}{K}(R))^\Gamma & \longrightarrow & \underset{\sim}{K}^\Gamma(\tilde{X},R)^\Gamma \\
\varepsilon^\Gamma\downarrow & & \downarrow \\
{}^b\underset{\sim}{h}^{\ell f}(\tilde{X},\underset{\sim}{K}(R))^{h\Gamma} & \longrightarrow & \underset{\sim}{K}^\Gamma(\tilde{X},R)^{h\Gamma}
\end{array}$$

The lower left hand vertical arrow is an equivalence by the second of the properties of ${}^b\underset{\sim}{h}^{\ell f}(-,\underset{\sim}{K}(R))$. We recall that if $X \xrightarrow{f} Y$ is an equivariant map, which is an equivalence without reference to the Γ–action, then $X^{h\Gamma} \to Y^{h\Gamma}$ is an equivalence; this is a general property of homotopy fixed point sets. Therefore, if we show that the map ${}^b\underset{\sim}{h}^{\ell f}(-,\underset{\sim}{K}(R)) \to \underset{\sim}{K}(\tilde{X},R)$ is an equivalence, the lowest horizontal arrow above will also be an equivalence.

Therefore, so is the composite

$$B\Gamma_+ \wedge \underset{\sim}{K}(R) \to \underset{\sim}{K}(R[\Gamma]) \to \underset{\sim}{K}^{\Gamma}(\tilde{X}, R)^{h\Gamma} ,$$

and the assembly map can be identified up to homotopy with the inclusion of a wedge summand of spectra. We now state our two main theorems. Let $\underset{\sim}{\mathcal{K}}(R)$ denote the non–connective Gersten–Wagoner spectrum of R; its homotopy groups agree with those of $\underset{\sim}{K}(R)$ in nonnegative dimensions, and in all dimensions if R is regular (see [22] or [46]). In III.20 a natural transformation ${}^b\underset{\sim}{h}^{\ell f}(-, \underset{\sim}{\mathcal{K}}(R)) \to \underset{\sim}{\mathcal{K}}(X, R)$ is constructed where $\underset{\sim}{\mathcal{K}}(X, R)$ denotes a non–connective version of $\underset{\sim}{K}(X, R)$; it also agrees with $\underset{\sim}{K}(X, R)$ in non–negative dimensions.

Theorem A. *Suppose X is a Riemannian manifold with $\pi_1(X) \cong \Gamma$, and $\tilde{X}$ is contractible. Equip $\tilde{X}$ with the Riemannian metric induced from X. If the map ${}^b\underset{\sim}{h}^{\ell f}(\tilde{X}, \underset{\sim}{\mathcal{K}}(R)) \to \underset{\sim}{\mathcal{K}}(\tilde{X}, R)$ is an equivalence, then the assembly map can be identified up to homotopy with the inclusion of a wedge summand of the spectrum $\underset{\sim}{\mathcal{K}}(R[\Gamma])$.*

The condition that ${}^b\underset{\sim}{h}^{\ell f}(\tilde{X}, \underset{\sim}{\mathcal{K}}(R)) \to \underset{\sim}{\mathcal{K}}(\tilde{X}, R)$ is an equivalence depends only on the behavior of the metric on $\tilde{X}$ "in the large." After some contemplation, one can see that it only depends on the structure (again in the large) of the metric space whose points are elements of Γ and where the metric is the one associated to a length function for some finite generating set for Γ. It does not depend on the fine algebraic structure of Γ; for instance, all torsion–free cocompact subgroups of the groups of isometries of Euclidean k–space give the same result.

A second result that we prove in this paper is that if G is a connected Lie group, K is a maximal compact subgroup, and G/K is equipped with a left–invariant Riemannian metric, then ${}^b\underset{\sim}{h}^{\ell f}(G/K, \underset{\sim}{\mathcal{K}}(R)) \to \underset{\sim}{\mathcal{K}}(G/K, R)$ is an equivalence. Since $\Gamma \backslash G/K$ is a Riemannian manifold, with G/K contractible, the following now is a consequence of Theorem A.

Theorem B. *Suppose Γ is a discrete, cocompact, torsion–free subgroup of a connected Lie group G. Then the assembly map*

$$B\Gamma_+ \wedge \underset{\sim}{\mathcal{K}}(R) \to \underset{\sim}{\mathcal{K}}(R[\Gamma])$$

may be identified up to homotopy with the inclusion of a wedge summand of spectra.