1
Euclidean Laplacian

1.1 The Laplacian

A fundamental aspect of scattering theory, and one to which I shall give considerable emphasis, is the parametrization of the continuous spectrum of differential operators, especially the Laplace operator. I therefore want to begin these lectures with a discussion of the spectral theory of the flat Laplacian on Euclidean space:

$$\Delta = D_1^2 + D_2^2 + \cdots + D_n^2 \text{ on } \mathbb{R}^n, \ D_j = \frac{1}{i}\frac{\partial}{\partial z_j} \tag{1.1}$$

where $z_1, \ldots, z_n$ are the standard coordinates. Notice that this is the 'geometer's[1] Laplacian' whereas the 'analyst's Laplacian' $-\Delta$.[2]

To a large extent below, except where it is really important, I shall avoid functional analytic statements relating to the boundedness of operators on Hilbert spaces. Thus I shall consider, at least initially, Δ as an operator on Schwartz' space[3] of $\mathcal{C}^\infty$ functions which decrease rapidly at infinity with all derivatives:

$$\mathcal{S}(\mathbb{R}^n) = \left\{ u : \mathbb{R}^n \longrightarrow \mathbb{C};\ \sup_{z\in\mathbb{R}^n} |z^\alpha D^\beta u(z)| < \infty \right\}. \tag{1.2}$$

[1] Of course it depends on the sort of 'geometer' you know; this positive Laplacian is the 0-form case of the Hodge Laplacian. Some geometers use the analysts' convention.

[2] The 'scattering theorist's Laplacian' is either $-i\Delta$ or $A = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta & 0 \end{pmatrix}$. The reason for considering A should become clearer in Section 3.2.

[3] See [42], Definition 7.1.2. It is somewhat contradictory to be using $\mathcal{S}(\mathbb{R}^n)$, which is a more subtle space topologically than are Hilbert spaces such as $L^2(\mathbb{R}^n)$; nevertheless doing so avoids the discussion of unbounded operators. See also [103].

The Fourier transform

$$\hat{f}(\zeta) = \int_{\mathbb{R}^n} e^{-iz\cdot\zeta} f(z) dz \tag{1.3}$$

is an endomorphism[4] of $\mathcal{S}(\mathbb{R}^n)$ with inverse

$$f(z) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iz\cdot\zeta} \hat{f}(\zeta) d\zeta. \tag{1.4}$$

Since[5] $\widehat{D_j f} = \zeta_j \hat{f}$, conjugation by the Fourier transform reduces any constant coefficient operator to multiplication by a function, in particular

$$\widehat{\Delta f} = |\zeta|^2 \hat{f}, \ \forall\ f \in \mathcal{S}(\mathbb{R}^n). \tag{1.5}$$

1.2 Spectral resolution

Using (1.5), and the inversion formula (1.4), the form of the spectral resolution [6] of Δ can be readily deduced. Introducing polar coordinates, $\zeta = \lambda\omega$, $\lambda = |\zeta|$ in (1.4) gives

$$f(z) = (2\pi)^{-n} \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{i\lambda z\cdot\omega} \lambda^{n-1} \hat{f}(\lambda\omega) d\omega d\lambda. \tag{1.6}$$

This can be rewritten as a decomposition of the identity operator:

(1.7)

$$\mathrm{Id} = \int_0^\infty E_0(\lambda) d\lambda,\ \ E_0(\lambda) f = (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} e^{i\lambda z\cdot\omega} \lambda^{n-1} \hat{f}(\lambda\omega) d\omega.$$

4 See [42], Theorem 7.1.5.
5 See [42], Lemma 7.1.4.
6 See [98] for a discussion of the spectral theorem; it is not necessary to know this result to proceed (in fact this admonition could be appended to many subsequent comments).

Here[7] $E_0(\lambda)d\lambda$ is a projection-valued measure[8] which gives the spectral decomposition of the Laplacian

$$\Delta = \int_0^\infty \lambda^2 E_0(\lambda)d\lambda. \tag{1.8}$$

The operator $E_0(\lambda)$ has range in the null space of $\Delta - \lambda^2$; as follows from the fact that the 'plane waves' $\Phi_0(z,\omega,\lambda) = \exp(i\lambda z\cdot\omega)$ are, for $\omega \in \mathbb{S}^{n-1}$, solutions of $(\Delta - \lambda^2)\Phi_0 = 0$. It is convenient to decompose $E_0(\lambda)$ as a product of two operators. Define[9]

$$(\Phi_0(\lambda)g)(z) = \int_{\mathbb{S}^{n-1}} \Phi_0(z,\omega,\lambda)g(\omega)d\omega,\ \Phi_0(z,\omega,\lambda) = e^{i\lambda z\cdot\omega}. \tag{1.9}$$

Thus $\Phi_0(\lambda) : \mathcal{C}^\infty(\mathbb{S}^{n-1}) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions. The formal adjoint operator is just[10]

$$(\Phi_0^*(\lambda)f)(\omega) = \int_{\mathbb{R}^n} \Phi_0(z,\omega,-\lambda)f(z)dz,\ \Phi_0^*(\lambda) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{S}^{n-1}), \tag{1.10}$$

since $\Phi_0(z,\omega,-\lambda) = \overline{\Phi_0(z,\omega,\lambda)}$. Then the definition (1.7) becomes

$$E_0(\lambda) = (2\pi)^{-n}\lambda^{n-1}\Phi_0(\lambda)\Phi_0^*(\lambda),\ \lambda > 0. \tag{1.11}$$

Now, for fixed $0 \neq \lambda \in \mathbb{R}$, $\Phi_0^*(\lambda)$ is surjective as a map (1.10).[11] Thus to compute the range of $E_0(\lambda)$ it is only necessary to find the range of $\Phi_0(\lambda)$. In fact it is as large as could reasonably be expected.

7 λ is the 'frequency' of the wave $e^{i\lambda z\cdot\omega}$.

8 It is not the case that $E_0(\lambda)$ maps $f \in \mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$; the form of the range is discussed below. One might therefore wonder on what space this is supposed to be a projection! One way to explain this is in terms of the average of the $E_0(\lambda)$. If $q \in \mathcal{C}_c^\infty((0,\infty))$ is a smooth function of compact support set $E_0(q)f = \int_0^\infty q(\lambda)E_0(\lambda)fd\lambda$. Then $E_0(q) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ and for any two functions $q, q' \in \mathcal{C}_c^\infty((0,\infty))$ it is always the case that $E_0(q)\circ E_0(q') = E_0(qq')$.

9 As a general convention I use the same notation for the operator and its Schwartz kernel. Of course this is a possible source of confusion and error; in particular one has to be careful as to which variables are regarded as parameters and what is the splitting into 'incoming' and 'outgoing' variables. Nevertheless I feel that this danger is outweighed by the consequent reduction in the number of symbols.

10 So one can reasonably say that $\Phi_0^*(z,\omega,\lambda) = \overline{\Phi^\dagger(z,\omega,\lambda)}$, where the $\dagger$ tells one to reverse the order of the variables and so get the transpose.

11 As follows from the properties of the Fourier transform, since any smooth function on the sphere $|\zeta| = \lambda > 0$ is the restriction of an element of $\mathcal{S}(\mathbb{R}^n)$.

Lemma 1.1 [12] *For $0 \neq \lambda \in \mathbb{R}$ the range of $\Phi_0(\lambda)$, acting on distributions on $\mathbb{S}^{n-1}$, is the null space of $\Delta - \lambda^2$ acting on the space, $\mathcal{S}'(\mathbb{R}^n)$, of tempered distributions on $\mathbb{R}^n$.*

1.3 Scattering matrix

Thus all the solutions of $(\Delta - \lambda^2)u = 0$ with u 'of polynomial growth' are superpositions of the elementary plane wave solutions $\Phi_0(z, \omega, \lambda) = e^{i\lambda z\cdot\omega}$ where $\omega \in \mathbb{S}^{n-1}$. The plane waves give a 'continuous[13] parametrization' of the eigenspace; there is a related 'functional parametrization' of it which is also important.

If g in (1.9) is taken to be smooth then the principle of stationary phase [14] can be used to understand the behaviour of $\Phi_0(\lambda)g$ as $|z| \to \infty$. Writing $z = |z|\theta$, $\theta = z/|z| \in \mathbb{S}^{n-1}$ gives

$$\Phi_0(\lambda)g(|z|\theta) = \int_{\mathbb{S}^{n-1}} e^{i|z|\lambda\theta\cdot\omega} g(\omega)d\omega. \tag{1.12}$$

The phase function $\theta \cdot \omega$ as a function of $\omega \in \mathbb{S}^{n-1}$ is stationary, i.e. has vanishing gradient, exactly at the two points $\omega = \pm\theta$. Since the Hessian at these points is non-degenerate[15] the stationary phase lemma gives a complete asymptotic expansion[16]

$$\begin{aligned}\Phi_0(\lambda)g(|z|\theta) \sim\ & e^{i\lambda|z|}(\lambda|z|)^{-\frac{1}{2}(n-1)} e^{-\frac{1}{4}\pi(n-1)i}(2\pi)^{\frac{1}{2}(n-1)} \sum_{j\geq 0} |z|^{-j} h_j^+(\theta) \\ & + e^{-i\lambda|z|}(\lambda|z|)^{-\frac{1}{2}(n-1)} e^{\frac{1}{4}\pi(n-1)i}(2\pi)^{\frac{1}{2}(n-1)} \sum_{j\geq 0} |z|^{-j} h_j^-(\theta),\ \lambda > 0,\end{aligned} \tag{1.13}$$

[12] This is simple to prove using the structure theory of distributions. Namely if $(\Delta - \lambda^2)u = 0$ with $u \in \mathcal{S}'(\mathbb{R}^n)$, the dual space to $\mathcal{S}(\mathbb{R}^n)$, then the Fourier transform $\widehat{u}(\zeta)$ satisfies $(|\zeta|^2 - \lambda^2)\widehat{u}(\zeta) = 0$. If $\lambda \neq 0$ it follows that, written in terms of polar coordinates $z = r\theta$, $\widehat{u} = \delta(r - |\lambda|)g'(\theta)$ for some distribution on the sphere $g' \in \mathcal{C}^{-\infty}(\mathbb{S}^{n-1})$. The inverse Fourier transform then shows that u is $\Phi_0(\lambda)g$ for $g = (2\pi)^{-n}\lambda^{n-1}g'$.

[13] Really this is a smooth parametrization. One view of scattering theory is that it describes the smoothness of the spectrum of appropriate operators.

[14] By a 'principle' here is meant an old theorem which has had many manifestations. For a precise statement of an appropriate version see [42], Section 7.7.

[15] That is, $\omega \cdot \theta$ is a Morse function on the sphere.

[16] This means that for any integer N the difference between the left side and the partial sum over $j \leq N$ on the right side is bounded, in $|z| \geq 1$, by $C|z|^{-N-1-\frac{1}{2}(n-1)}$ for some constant C. The power here is just the size of the first term dropped from the sums. In fact the same is true after any number of formal derivatives with respect to θ, or $r = |z|$, are taken (on both sides of course).

in which $h_0^\pm(\theta) = g(\pm\theta)$ and the $h_j^\pm$ for $j \ge 1$ are all given by polynomials in the Laplacian on the sphere applied to $g(\pm\theta)$.

Lemma 1.2 [17] *For each $\lambda > 0$ and each $h \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ there is a unique solution to $(\Delta - \lambda^2)u = 0$ such that as $|z| \to \infty$*[18]

$$(1.15)\quad \begin{aligned} u(|z|\theta) &= e^{i\lambda|z|}|z|^{-\frac{1}{2}(n-1)}h(\theta) \\ &\quad + e^{-i\lambda|z|}|z|^{-\frac{1}{2}(n-1)}h'(\theta) + O\left(|z|^{-\frac{1}{2}(n+1)}\right) \end{aligned}$$

where $h' \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, and necessarily[19]

$$(1.16)\quad h'(\theta) = A_0 h(\theta) = i^{n-1}h(-\theta).$$

This parametrizes the generalized eigenspace with eigenvalue λ^2 by the distributions[20] on the sphere at infinity. Notice that $\pm\lambda$ give different parametrizations of the same space, one in terms of h and the other in terms of h'. The relationship between these two parametrizations is given by (1.16) and this operator, mapping $h(\theta)$ to $i^{(n-1)}h(-\theta)$, is the 'absolute scattering matrix' for Euclidean space. It is a unitary isomorphism of $\mathcal{C}^\infty(\mathbb{S}^{n-1})$. [21]

There are various stronger forms of this lemma, as far as the uniqueness part is concerned. One particularly convenient one arises from the

[17] The existence part follows from (1.13). To prove the uniqueness it is only necessary to prove a variant of (1.13) for $\Phi_0(\lambda)g$ where $g \in \mathcal{C}^{-\infty}(\mathbb{S}^{n-1})$ is a distribution. This can be done by using the same formula, (1.12), integrated against a test function in $\mathcal{C}^\infty(\mathbb{S}^{n-1})$. The stationary phase expansion in the θ variable shows that

$$(1.14)\quad \begin{aligned} \Phi_0(\lambda)g(|z|\theta) &= e^{i\lambda|z|}\,(\lambda|z|)^{-\frac{1}{2}(n-1)}\,e^{-\frac{1}{4}\pi(n-1)i}(2\pi)^{\frac{1}{2}(n-1)}g(\theta) \\ &\quad + e^{-i\lambda|z|}\,(\lambda|z|)^{-\frac{1}{2}(n-1)}\,e^{\frac{1}{4}\pi(n-1)i}(2\pi)^{\frac{1}{2}(n-1)}g(-\theta) + u' \end{aligned}$$

where $u' \in H^{-\infty}(\mathbb{R}^n)$ is in the union of all the standard Sobolev spaces; moreover g is determined by this expansion since neither the first two terms separately, not their sum, can be in $H^{-\infty}(\mathbb{R}^n)$ unless $g = 0$. Given two solutions of the form (1.15), the difference is a solution with $h = 0$. From Lemma 1.1 it follows that this difference is of the form $\Phi_0(\lambda)g$ for some g. The uniqueness of the expansion (1.14) then shows that $g = 0$. Some further comments on the uniqueness will be made in Lecture 2.

[18] The 'big Oh' notation here means that the difference of the left and right sides is bounded by $C|z|^{-\frac{1}{2}(n+1)}$ in $|z| \ge 1$ for some constant C.

[19] As defined here the operator A_0 is independent of λ. However, there is also a unique solution on the form (1.15) for $\lambda < 0$. If n is odd, the resulting operator mapping h to h' is the same. If n is even it is not, rather it is $-A_0$.

[20] I mean here that the map $h \mapsto u \in \mathcal{S}'(\mathbb{R}^n)$ extends by continuity to all $h \in \mathcal{C}^{-\infty}(\mathbb{S}^{n-1})$ and then gives a parametrization of all the tempered generalized eigenfunctions.

[21] If n is odd it is an involution, i.e. $A_0 \circ A_0 = \mathrm{Id}$, whereas if n is even it is a fourth root of unity in the sense that $A_0^4 = \mathrm{Id}$. This sort of behaviour, depending on the parity of the dimension, can be seen much more strongly in 1.6.

observation that the function $|z|^{-\frac{1}{2}(n-1)}$ is locally square-integrable near 0 and that $|z|^{-\frac{1}{2}(n+1)}$ is square-integrable near $|z| = \infty$.[22] Thus (1.15) implies that

$$u(|z|\theta) = e^{i\lambda|z|}|z|^{-\frac{1}{2}(n-1)}h(\theta) + e^{-i\lambda|z|}|z|^{-\frac{1}{2}(n-1)}h'(\theta) + u',\ u' \in L^2(\mathbb{R}^n). \tag{1.17}$$

Conversely, for a solution to $(\Delta - \lambda^2)u = 0$, this implies (1.15) and hence (1.13).

Notice from (1.13) that the map $\mathcal{C}^\infty(\mathbb{S}^{n-1}) \longmapsto \mathcal{S}'(\mathbb{R}^n)$ which gives the unique solution of the form (1.15) is

$$u(z) = P_0(\lambda)h = \lambda^{\frac{1}{2}(n-1)}e^{\frac{1}{4}\pi(n-1)i}(2\pi)^{-\frac{1}{2}(n-1)}\Phi_0(\lambda)h,\ \lambda > 0. \tag{1.18}$$

It would be reasonable to call the operator $P_0(\lambda)$ the 'Poisson operator' for the 'boundary problem' which seeks the solution to $(\Delta - \lambda^2)u = 0$ of the form (1.15) with h given.[23]

1.4 Resolvent family

I should pay at least lip service to the fundamental fact that the Laplacian is an essentially self-adjoint operator.[24] In particular the inverse of the operator $\Delta - \sigma$, for $\sigma \in \mathbb{C}\setminus\mathbb{R}$ is a bounded operator on $L^2(\mathbb{R}^n)$. This is certainly true and much more can be seen, namely that this operator can be obtained in terms of the Fourier transform:

$$(\Delta - \sigma)^{-1}f(z) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{iz\cdot\zeta}(|\zeta|^2 - \sigma)^{-1}\hat{f}(\zeta)d\zeta \tag{1.19}$$

whenever $\sigma \in \mathbb{C} \setminus [0,\infty)$.[25]

Since the spectrum[26] is confined to the positive real axis it is convenient to introduce $\lambda^2 = \sigma$ as a modified spectral parameter. There are two obvious normalizations of the choice of λ; I shall choose the 'physical

[22] That is, the function is square-integrable on the complement of any ball of positive radius around the origin.
[23] The mapping properties of an operator such as $P_0(\lambda)$ can be understood in terms of Besov spaces, see [43].
[24] If you want to know what this means see [98].
[25] Since then $|\zeta|^2 - \sigma$ has no zeroes for $\zeta \in \mathbb{R}^n$.
[26] The spectrum is the singular set of the resolvent family .

domain' to be the set[27]

$$\mathcal{P} = \{\lambda \in \mathbb{C}; \operatorname{Im}\lambda < 0\}. \tag{1.20}$$

Then define

$$R_0(\lambda) = (\Delta - \lambda^2)^{-1}, \ \lambda \in \mathcal{P}, \text{ i.e. } \operatorname{Im}\lambda < 0. \tag{1.21}$$

I will usually refer to this, slightly incorrectly,[28] as 'the resolvent.' From (1.19) it follows that

$$R_0(\lambda) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \text{ for } \operatorname{Im}\lambda < 0. \tag{1.22}$$

It is the unique operator with this property such that $(\Delta - \lambda^2) \circ R_0(\lambda) = \operatorname{Id}$ on $\mathcal{S}(\mathbb{R}^n)$.

1.5 Limiting absorption principle

The resolvent of the Laplacian can be written as an integral operator:

$$\begin{aligned} R_0(\lambda)f(z) &= \int_{\mathbb{R}^n} R_0(\lambda, z, z')f(z')dz', \\ R_0(\lambda, z, z') &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta} \frac{d\zeta}{(|\zeta|^2 - \lambda^2)}. \end{aligned} \tag{1.23}$$

The integral here is not absolutely convergent.[29] To avoid worrying about this[30] I shall consider instead the kth power of the resolvent, where for $k > \frac{1}{2}n$ the corresponding integral *is* absolutely convergent

$$R_0^k(\lambda, z, z') = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta} \frac{d\zeta}{(|\zeta|^2 - \lambda^2)^k}, \ \operatorname{Im}\lambda < 0. \tag{1.27}$$

[27] In the lectures themselves I used the opposite convention, that $\operatorname{Im}\lambda > 0$ in the physical domain, I regretted it then .... I hope that all the sign errors have been eliminated, but I am not too confident.

[28] In that the resolvent is $(\Delta - \sigma)^{-1}$.

[29] It is relatively straightforward to compute the form of these kernels 'explicitly'; the result (as with almost everything else) is simpler in the odd-dimensional case than the even-dimensional one. If $n = 1$ then

$$R_0(\lambda, z, z') = \lambda^{-1} \exp(-i\lambda|z - z'|). \tag{1.24}$$

If $n \geq 3$ is odd then there is a polynomial, q_n, of degree $\frac{1}{2}(n-1)$ in one variable such that

$$R_0(\lambda; z, z') = |z - z'|^{-n+2} q_n(\lambda|z - z'|) \exp(i\lambda|z - z'|). \tag{1.25}$$

For $n \geq 2$ even

$$R_0(\lambda; z, z') = \frac{1}{4i} \left(\frac{\lambda}{2\pi|z - z'|}\right)^{\frac{1}{2}n-1} \operatorname{Ha}^{(1)}_{\frac{1}{2}n-1}(\lambda|z - z'|) \tag{1.26}$$

where $\operatorname{Ha}^{(1)}_j(z)$ is a Hankel function.

[30] Not that it is a serious problem.

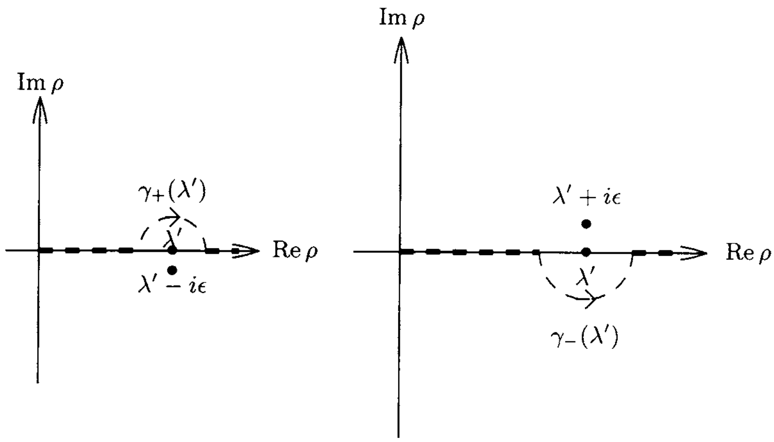

Fig. 1. The contours $\gamma_+(\lambda')$ and $\gamma_-(\lambda')$.

I am especially interested in what happens as $\operatorname{Im}\lambda \uparrow 0$ and the spectral parameter $\sigma = \lambda^2$ approaches $[0, \infty)$. Introducing polar coordinates in ζ, as in (1.7), but now writing $\zeta = \rho\omega$, gives

$$(1.28) \qquad R_0^k(\lambda, z, z') = (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} \int_0^\infty e^{i\rho(z-z')\cdot\omega} \frac{\rho^{n-1}d\rho}{(\rho^2 - \lambda^2)^k} d\omega.$$

The integrand is holomorphic in ρ away from $\rho = \pm\lambda$, where there is a pole. If $\lambda = \lambda' - i\epsilon$ with $\lambda' > 0$ and $\epsilon > 0$ and small, then Cauchy's theorem can be used to move the contour in (1.28) to $\gamma_+(\lambda')$ as in Figure 1:

(1.29)
$$R_0^k(\lambda, z, z') = (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} \int_{\gamma_+(\lambda')} e^{i\rho(z-z')\cdot\omega} \frac{\rho^{n-1}d\rho}{(\rho^2 - \lambda^2)^k} d\omega, \ \lambda = \lambda' - i\epsilon.$$

Now the limit as $\epsilon \downarrow 0$ in (1.29) is not singular, provided $\lambda' > 0$. If $\lambda = -\lambda' - i\epsilon$ where λ' is still positive then in place of (1.29)

(1.30)
$$R_0^k(\lambda, z, z') = (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} \int_{\gamma_-(\lambda')} e^{i\rho(z-z')\cdot\omega} \frac{\rho^{n-1}d\rho}{(\rho^2 - \lambda^2)^k} d\omega,$$
$$\lambda = -\lambda' - i\epsilon,$$

with $\gamma_-(\lambda')$ the contour going 'the other way' around λ'. Again the limit as $\epsilon \downarrow 0$ can be taken. Since the two limiting points $\pm\lambda'$ correspond to the same point $\sigma = (\lambda')^2$ in the spectrum it is natural to consider the difference:

$$R_0^k(\lambda, z, z') - R_0^k(-\lambda, z, z') = \\ (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} \int_{\gamma_+(\lambda)-\gamma_-(\lambda)} e^{i\rho(z-z')\cdot\omega} \frac{\rho^{n-1}d\rho}{(\rho^2-\lambda^2)^k} d\omega, \ \lambda > 0. \tag{1.31}$$

The difference of the two contours is homotopic to a clockwise circle of small radius around the single point λ, so Cauchy's theorem can be used to evaluate the integral as a residue. Using the identity $(\Delta - \lambda^2)^j R_0^k = R_0^{k-j}(\lambda)$ for $k > j$

$$R_0(\lambda; z, z') - R_0(-\lambda; z, z') \\ = \frac{1}{2i}(2\pi)^{-(n-1)}\lambda^{n-2} \int_{\mathbb{S}^{n-1}} e^{i\lambda(z-z')\cdot\omega} d\omega, \ \lambda > 0. \tag{1.32}$$

This is the limiting absorption principle or it can be called, perhaps more correctly, Stone's theorem. [31]

1.6 Analytic continuation

Formulæ (1.29) and (1.30) make sense for ϵ small compared to the radius of the circular part of the contour (and hence with respect to λ'), of either sign. This shows that $R_0(\lambda, z, z')$ can be continued analytically, as a function of λ, through the real axis, at least away from 0. Indeed if I define $M(\lambda)$, in terms of right side of (1.32):

$$M(\lambda, z, z') = \frac{1}{2i}(2\pi)^{-(n-1)} \int_{\mathbb{S}^{n-1}} e^{i\lambda(z-z')\cdot\omega} d\omega, \ \lambda > 0, \tag{1.34}$$

then observe that $M(\lambda; z, z')$ extends to be an entire function of $\lambda \in \mathbb{C}$. I shall denote, temporarily, by $\widetilde{R}_0(\lambda)$ the function[32] defined by analytic continuation of $R_0(\lambda)$ across $(0, \infty)$. Thus $\widetilde{R}_0(\lambda) = R_0(\lambda)$ in $\operatorname{Im}\lambda < 0$,

[31] Which can be stated briefly in the present context as the assertion that the spectral resolution can be obtained from the difference of the limits, from above and below, of the resolvent family on the spectrum. Notice that by inserting the Fourier transform in (1.7) it follows that

$$E_0(\lambda) = \frac{\lambda i}{\pi}\left(R_0(\lambda) - R_0(-\lambda)\right), \ \lambda \in (0, \infty). \tag{1.33}$$

[32] Really to be thought of as an operator.

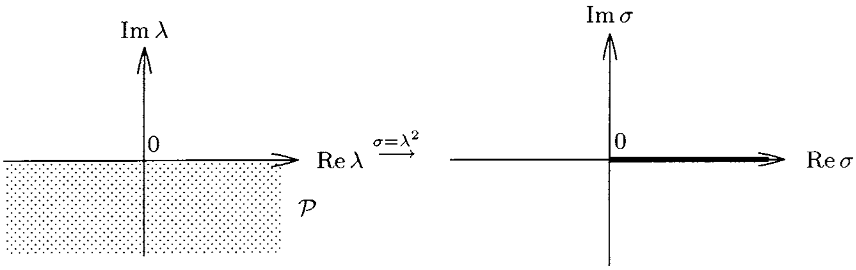

Fig. 2. Analytic continuation of the resolvent for n odd.

but $\widetilde{R}_0(\lambda)$ is also defined near the positive real axis. From (1.32) it follows that, for λ near the positive real axis with $\operatorname{Im}\lambda > 0$,

$$\widetilde{R}_0(\lambda) = R_0(-\lambda) + \lambda^{n-2}M(\lambda). \tag{1.35}$$

Thus in fact $\widetilde{R}_0(\lambda)$ extends to be holomorphic for all $\operatorname{Im}\lambda > 0$ as well as near $(0,\infty)$ and in $\operatorname{Im}\lambda < 0$, i.e. $\widetilde{R}_0(\lambda)$ is holomorphic in $\mathbb{C}\setminus(-\infty,0]$.

Using the antipodal map in the sphere it follows from (1.34) that

$$M(-\lambda) = M(\lambda). \tag{1.36}$$

Applying (1.35) twice gives

$$\lim_{\epsilon\downarrow 0}\widetilde{R}_0(-\lambda'+i\epsilon) - \lim_{\epsilon\uparrow 0}\widetilde{R}_0(-\lambda'+i\epsilon) = \begin{cases} 0 & n \text{ odd} \\ 2(\lambda')^{n-2}M(\lambda') & n \text{ even} \end{cases} \quad \lambda' > 0, \tag{1.37}$$

This shows the basic difference between the odd- and even-dimensional cases. For odd $n \ge 3$, the resolvent kernel is locally integrable in z, z' and entire[33] as a function of λ; (see Figure 2) for $n = 1$ it is meromorphic in λ with a simple pole at 0. In the even-dimensional case a similar result is valid, except that the kernel only extends to be entire on the logarithmic covering of the complex plane, Λ, i.e. as a function of the variable $\log\lambda$ (see Figure 3).[34] Thus if $R_0^\flat(\tau) = R_0(\lambda)$ the 'physical domain' for $R_0^\flat(\tau)$ can be taken as $\{\tau \in \mathbb{R}\times i(-\pi,0) \subset \mathbb{C}\}$ and then $R_0^\flat(\tau)$ extends to

[33] It follows from (1.23) that there is neither an essential singularity, nor a pole, at $\lambda = 0$.

[34] In the even dimensional case the behaviour as $\lambda \to 0$ can also be analyzed; in fact

$$R_0(\lambda) = R_0'(\lambda) + M(\lambda)\lambda^{n-2}\log\lambda \tag{1.38}$$

where $R_0'(\lambda)$ is entire.