I

THE PICARD GROUP AND THE RIEMANN–ROCH THEOREM

Unless otherwise stated, we consider surfaces with their Zariski topology (the closed subsets are the algebraic subvarieties); 'sheaf' will mean 'coherent algebraic sheaf'. This is a matter of convention: Serre's general theorems ([GAGA]) give a bijection between algebraic and analytic coherent sheaves which preserves exactness, cohomology, etc. All our arguments with coherent algebraic sheaves will remain valid in the analytic context.

Fact I.1 The Picard group

Let S be a smooth variety. Recall that the Picard group of S, $\operatorname{Pic} S$, is the group of isomorphism classes of invertible sheaves (or of line bundles) on S. To every effective divisor D on S there corresponds an invertible sheaf $\mathcal{O}_S(D)$ and a section $s \in H^0(\mathcal{O}_S(D))$, $s \neq 0$, which is unique up to scalar multiplication, such that $\operatorname{div}(s) = D$. We define $\mathcal{O}_S(D)$ for an arbitrary D by linearity. The map $D \mapsto \mathcal{O}_S(D)$ identifies $\operatorname{Pic} S$ with the group of linear equivalence classes of divisors on S.

Let X be another smooth variety and $f : S \to X$ a morphism. We can define the inverse image with respect to f of an invertible sheaf, to get a homomorphism $f^* : \operatorname{Pic} X \to \operatorname{Pic} S$. If f is surjective, then we can also define the inverse image of a divisor, in such a way that $f^*\mathcal{O}_X(D) = \mathcal{O}_S(f^*(D))$: just note that the inverse image of a non-zero section of $\mathcal{O}_X(D)$ is non-zero.

If f is a morphism of surfaces which is generically finite of degree d, then we define the direct image f_*C of an irreducible curve C by setting

$f_*C = 0$	if $f(C)$ is a point,
$f_*C = r\Gamma$	if $f(C)$ is a curve Γ, the morphism $C \to \Gamma$ induced by f being finite of degree r.

We define f_*D for all divisors D on S by linearity. One can check immediately that $D \equiv D'$ implies $f_*D \equiv f_*D'$. It follows from the definition that

$$f_*f^*D = dD \quad \text{for all divisors } D \text{ on } S \ .$$

The particular importance of the Picard group in the case of surfaces stems from the existence of an intersection form, which we now define.

Definition I.2 *Let C, C' be two distinct irreducible curves on a surface S, $x \in C \cap C'$, $\mathcal{O}_x$ the local ring of S at x. If f (resp. g) is an equation of C (resp. C') in $\mathcal{O}_x$, the intersection multiplicity of C and C' at x is defined to be*

$$m_x(C \cap C') = \dim_{\mathbb{C}} \mathcal{O}_x/(f,g) \ .$$

By the Nullstellensatz the ring $\mathcal{O}_x/(f,g)$ is a finite-dimensional vector space over $\mathbb{C}$. The reader should confirm that this definition corresponds to the intuitive notion of intersection number. For example, we see that $m_x(C \cap C') = 1$ if and only if f and g generate the maximal ideal $_x$, i.e. form a system of local coordinates in a neighbourhood of x : C and C' are then said to be transverse at x.

Definition I.3 *If C, C' are two distinct irreducible curves on S, the intersection number $(C.C')$ is defined by:*

$$(C.C') = \sum_{x \in C \cap C'} m_x(C \cap C') \ .$$

Recall that the ideal sheaf defining C (resp. C') is just the invertible sheaf $\mathcal{O}_S(-C)$ (resp. $\mathcal{O}_S(-C')$); define

$$\mathcal{O}_{C \cap C'} = \mathcal{O}_S/(\mathcal{O}_S(-C) + \mathcal{O}_S(-C')) \ .$$

The sheaf $\mathcal{O}_{C \cap C'}$ is a skyscraper sheaf, concentrated at the finite set $C \cap C'$; at each of these points we have $(\mathcal{O}_{C \cap C'})_x = \mathcal{O}_x/(f,g)$ (with the notation of I.1). It is now clear that

$$(C.C') = \dim H^0(S, \mathcal{O}_{C \cap C'}) \ .$$

For any sheaf L on S, let $\chi(L) = \sum_i (-1)^i h^i(S,L)$ be the Euler–Poincaré characteristic of L.

Theorem I.4 *For L, L' in* Pic S, *define*

$$(L.L') = \chi(\mathcal{O}_S) - \chi(L^{-1}) - \chi(L'^{-1}) + \chi(L^{-1} \otimes L'^{-1}) \ .$$

Then (.) *is a symmetric bilinear form on* Pic S, *such that if* C *and* C' *are two distinct irreducible curves on* S *then*

$$(\mathcal{O}_S(C).\mathcal{O}_S(C')) = (C.C') .$$

Proof We start by proving the last equality.

Lemma I.5 *Let* $s \in H^0(S, \mathcal{O}_S(C))$ (*resp.* $s' \in H^0(S, \mathcal{O}_S(C'))$) *be a non-zero section vanishing on* C (*resp.* C'). *The sequence*

$$0 \to \mathcal{O}_S(-C-C') \xrightarrow{(s',-s)} \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-C') \xrightarrow{(s,s')} \mathcal{O}_S \longrightarrow \mathcal{O}_{C\cap C'} \to 0$$

is exact.

Proof Let $f, g \in \mathcal{O}_x$ be local equations for C, C' at a point $x \in S$; we must show that the sequence

$$0 \to \mathcal{O}_x \xrightarrow{(g,-f)} \mathcal{O}_x^2 \xrightarrow{(f,g)} \mathcal{O}_x \to \mathcal{O}_x/(f,g) \to 0$$

is exact, i.e. that if $a, b \in \mathcal{O}_x$ are such that $af = bg$, then there exists $k \in \mathcal{O}_x$ such that $a = kg$, $b = kf$.

This follows immediately, say from the fact that $\mathcal{O}_x$ is a factorial ring and f, g are coprime (otherwise C and C' would have a common component). The highbrow reader can use the (much weaker) fact that $\mathcal{O}_x$ is Cohen–Macaulay.

Lemma I.5 and the additivity of the Euler–Poincaré characteristic give $(\mathcal{O}_S(C).\mathcal{O}_S(C')) = (C.C')$. To prove the theorem it remains to show that (.) is a bilinear form on Pic S (the symmetry is obvious).

Lemma I.6 *Let* C *be a non-singular irreducible curve on* S. *For all* $L \in \operatorname{Pic} S$, *we have*

$$(\mathcal{O}_S(C).L) = \deg(L_{|C}) .$$

Proof The exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_S(-C) & \to & \mathcal{O}_S & \to & \mathcal{O}_C & \to & 0 \\ 0 & \to & L^{-1}(-C) & \to & L^{-1} & \to & L^{-1}\otimes\mathcal{O}_C & \to & 0 \end{array}$$

give the following relations between Euler–Poincaré characteristics:

$$\begin{aligned} \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) &= \chi(\mathcal{O}_C) \\ \chi(L^{-1}) - \chi(L^{-1}(-C)) &= \chi(L^{-1}_{|C}) \end{aligned}$$

whence

$$\begin{aligned}(\mathcal{O}_S(C).L) &= \chi(\mathcal{O}_C) - \chi(L^{-1}_{|C}) \\ &= -\deg L^{-1}_{|C} \quad \text{by the Riemann–Roch theorem on } C \\ &= \deg L_{|C}, \quad \text{which proves the lemma.}\end{aligned}$$

For L_1, L_2, $L_3 \in \operatorname{Pic} S$, consider the expression

$$s(L_1, L_2, L_3) = (L_1.L_2 \otimes L_3) - (L_1.L_2) - (L_1.L_3) \ .$$

It is clear by definition of the product that this is symmetric in L_1, L_2 and L_3; moreover Lemma I.6 shows $s(L_1, L_2, L_3)$ is zero when $L_1 = \mathcal{O}_S(C)$, with C a non-singular curve. Similarly, $s(L_1, L_2, L_3)$ is zero if L_2 or L_3 is of this form.

For the general case, we recall a theorem of Serre (cf. [FAC]):

Fact I.7 *Let D be a divisor on S, and H a hyperplane section of S (for a given embedding). There exists $n \geqslant 0$ such that $D + nH$ is a hyperplane section (for another embedding). In particular we can write $D \equiv A - B$, where A and B are smooth curves on S, with $A \equiv D + nH$ and $B \equiv nH$.*

Now let L, L' be any two invertible sheaves. We can write $L' = \mathcal{O}_S(A - B)$, where A and B are two smooth curves on S. Noting that $s(L, L', \mathcal{O}_S(B)) = 0$, we get

$$(L, L') = (L.\mathcal{O}_S(A)) - (L.\mathcal{O}_S(B)) \ .$$

Via Lemma I.6, we deduce that $(L.L')$ is linear in L. This completes the proof of Theorem I.4.

If D, D' are two divisors on S, we write $D.D'$ for $(\mathcal{O}_S(D).\mathcal{O}_S(D'))$. The point of Theorem I.4 is that we can calculate this product by replacing D (or D') by a linearly equivalent divisor. Here are two applications of this principle.

Proposition I.8

(i) *Let C be a smooth curve, $f : S \to C$ a surjective morphism, F a fibre of f. Then $F^2 = 0$.*

(ii) *Let S' be a surface, $g : S \to S'$ a generically finite morphism of degree d, D and D' divisors on S. Then $g^*D.g^*D' = d(D.D')$.*

Proof (i) $F = f^*[x]$, for some $x \in C$. There exists a divisor A on C, linearly equivalent to x, such that $x \notin A$, so that $F \equiv f^*A$. Since

f^*A is a linear combination of fibres of f all distinct from F, we have $F^2 = F.f^*A = 0$.

(ii) As in I.7, it is enough to prove the formula when D and D' are hyperplane sections of S (in two different embeddings). There exists an open set U in S' over which g is étale. We can move D and D' so that they meet transversely and their intersection lies in U. It is then clear that g^*D and g^*D' meet transversely and that $g^*D \cap g^*D' = g^{-1}(D \cap D')$; hence the result.

Examples I.9

(a) $S = \mathbb{P}^2$.
$\operatorname{Pic} \mathbb{P}^2 = \mathbb{Z}$: every curve of degree d is linearly equivalent to d times a line. Let C, C' be curves of degrees d, d', and L, L' distinct lines; since $C \equiv dL$, $C' \equiv d'L'$, Theorem I.4 gives Bezout's theorem:

$$C.C' = dL.dL' = dd'(L.L') = dd' .$$

(b) $S = \mathbb{P}^1 \times \mathbb{P}^1 =$ smooth quadric in $\mathbb{P}^3$ – because of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, defined by

$$((X,T);(X',T')) \mapsto (XX', XT', TX', TT').$$

Let $h_1 = \{0\} \times \mathbb{P}^1$, $h_2 = \mathbb{P}^1 \times \{0\}$, $U = (\mathbb{P}^1 \times \mathbb{P}^1) - h_1 - h_2$. The open set U is isomorphic to the affine space $\mathbb{A}^2$, so every divisor on U is the divisor of a rational function. Let D be a divisor on $\mathbb{P}^1 \times \mathbb{P}^1$; then $D_{|U} = \operatorname{div}(\phi)$ on U, so there exist integers n and m such that

$$D = \operatorname{div}(\phi) + mh_1 + nh_2$$

and $D \equiv mh_1 + nh_2$.

Thus $\operatorname{Pic} S$ is generated by the classes of h_1 and h_2. It is clear that $h_1.h_2 = 1$. To find h_1^2, by I.4, we can replace h_1 by the curve $C_\infty = \{\infty\} \times \mathbb{P}^1$ which is linearly equivalent to h_1; since $h_1 \cap C_\infty = \emptyset$, we have $h_1^2 = h_1.C_\infty = 0$. Similarly $h_2^2 = 0$. It follows that $\mathcal{O}_S(h_1)$ and $\mathcal{O}_S(h_2)$ generate $\operatorname{Pic} S$, and the intersection product is given by $h_1^2 = h_2^2 = 0$, $h_1.h_2 = 1$.

Let Γ be a curve in $\mathbb{P}^1 \times \mathbb{P}^1$; it is defined by a bihomogeneous equation, i.e. homogeneous of weight m in the coordinates (X,T) and of weight n in (X',T'); Γ is said to have bidegree (m,n), and

$\Gamma \equiv mh_1 + nh_2$. If Γ' is a curve of bidegree (m', n'), Theorem I.4 gives:

$$\Gamma.\Gamma' = (mh_1 + nh_2).(m'h_1 + n'h_2) = mn' + nm' \ .$$

I.10 Topological interpretation Here we use analytic sheaves on S, and the ordinary topology. We write ${}^h\mathcal{O}_S$ for the sheaf of holomorphic functions on S, considered as an analytic manifold.

Consider the exponential map $e : {}^h\mathcal{O}_S \to {}^h\mathcal{O}_S^*$, given by $e(f) = \exp(2\pi i f)$. It is locally surjective (local existence of logarithms), and its kernel clearly consists of locally constant integer-valued functions. In other words there is an exact sequence

$$0 \to \mathbb{Z} \to {}^h\mathcal{O}_S \xrightarrow{e} {}^h\mathcal{O}_S^* \to 1 \ .$$

Let us study the derived long exact cohomology sequence. Since $H^0(S, {}^h\mathcal{O}_S) = \mathbb{C}$ and $H^0(S, {}^h\mathcal{O}_S^*) = \mathbb{C}^*$, we can start with the H^1:

$$0 \to H^1(S,\mathbb{Z}) \to H^1(S, {}^h\mathcal{O}_S) \to H^1(S, {}^h\mathcal{O}_S^*) \to H^2(S,\mathbb{Z}) \to H^2(S, {}^h\mathcal{O}_S) \ .$$

We know that $H^1(S, \mathcal{O}_S^*)$ is canonically identified with $\operatorname{Pic} S$ (strictly speaking the analytic Pic, but from [GAGA] this is the same thing; likewise $H^i(S, {}^h\mathcal{O}_S) \cong H^i(S, \mathcal{O}_S)$). So the group $\operatorname{Pic} S$ appears as an extension:

$$0 \to T \to \operatorname{Pic} S \to NS(S) \to 0$$

of two groups which are different in nature: $T = H^1(S, \mathcal{O}_S)/H^1(S, \mathbb{Z})$ is a divisible group (Hodge theory shows that $H^1(S, \mathbb{Z})$ is a lattice in $H^1(S, \mathcal{O}_S)$, so T has a natural structure of complex torus); whereas $NS(S) \subset H^2(S, \mathbb{Z})$ is a finitely generated group, called the Néron–Severi group of S.

The map $c : \operatorname{Pic} S \to H^2(S, \mathbb{Z})$ can be described topologically as follows. If $C \subset S$ is an irreducible curve, the restriction $H^2(S, \mathbb{Z}) \to H^2(C, \mathbb{Z}) \cong \mathbb{Z}$ gives a linear form on $H^2(S, \mathbb{Z})$, hence by Poincaré duality an element $c(C) \in H^2(S, \mathbb{Z})$; we define $c(D)$ for any divisor D by linearity. Then $c(D).c(D') = D.D'$ for divisors D, D' on S; in other words, the intersection form comes from the bilinear form on $NS(S)$ induced by the cup product.

If f is a morphism from S to a smooth variety X, then $f^*c(S) = c(f^*D)$ for any divisor D on X. If X is a surface, the Gysin homomorphism $f_* : H^2(S, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is defined, and $f_*c(D) = c(f_*D)$ for any D on S.

Let us come back to our algebraic set-up. We recall without proof Serre duality:

Theorem I.11 *Let S be a surface, and L a line bundle on S. Let ω_S be the line bundle of differential 2-forms on S. Then $H^2(S,\omega_S)$ is a 1-dimensional vector space; for $0 \leqslant i \leqslant 2$, the cup-product pairing*

$$H^i(S,L) \otimes H^{2-i}(S,\omega_S \otimes L^{-1}) \to H^2(S,\omega_S) \xrightarrow{\sim} \mathbb{C}$$

defines a duality. In particular, $\chi(L) = \chi(\omega_S \otimes L^{-1})$.

We can now prove the Riemann–Roch theorem.

Theorem I.12 (Riemann–Roch) *For all $L \in \operatorname{Pic} S$,*

$$\chi(L) = \chi(\mathcal{O}_S) + \frac{1}{2}(L^2 - L.\omega_S) .$$

Proof Let us compute $(L^{-1}.L \otimes \omega_S^{-1})$. By definition of the intersection product

$$(L^{-1}.L \otimes \omega_S^{-1}) = \chi(\mathcal{O}_S) - \chi(L) - \chi(\omega_S \otimes L^{-1}) + \chi(\omega_S) .$$

By Serre duality, $\chi(\omega_S) = \chi(\mathcal{O}_S)$ and $\chi(\omega_S \otimes L^{-1}) = \chi(L)$, and hence

$$(L^{-1}.L \otimes \omega_S^{-1}) = 2(\chi(\mathcal{O}_S) - \chi(L)) .$$

Using the bilinearity of the intersection form gives at once the stated formula.

Remarks I.13

(i) We will usually be writing these two theorems in terms of divisors; set $h^i(D) = h^i(S,\mathcal{O}_S(D))$; furthermore, it is traditional to let K denote any divisor such that $\mathcal{O}_S(K) = \omega_S$, and to call K a 'canonical divisor'. Serre duality then gives $h^i(D) = h^{2-i}(K-D)$ for $0 \leqslant i \leqslant 2$; and the Riemann–Roch theorem takes the form

$$h^0(D) + h^0(K-D) - h^1(D) = \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - D.K) .$$

Usually we will not have any information about $h^1(D)$, and we will use Riemann–Roch as an inequality

$$H^0(D) + H^0(K-D) \geqslant \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - D.K) .$$

(ii) We have given the Riemann–Roch theorem in its classical form. Nowadays one usually takes the Riemann–Roch theorem to mean Hirzebruch's version, which is the conjunction of I.12 and of the important formula of M. Noether, which we will assume:

I.14 Noether's formula

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K_S^2 + \chi_{\text{top}}(S)) ,$$

where $\chi_{\text{top}}(S)$ *is the topological Euler–Poincaré characteristic of* S: $\chi_{\text{top}}(S) = \sum(-1)^i b_i$, with $b_i = \dim_{\mathbb{R}} H^i(S, \mathbb{R})$.

Here is an important consequence of the Riemann–Roch formula.

I.15 The genus formula *Let* C *be an irreducible curve on a surface* S. *The genus of* C, *defined by* $g(C) = h^1(C, \mathcal{O}_C)$, *is given by* $g(C) = 1 + \frac{1}{2}(C^2 + C.K)$.

Proof The exact sequence

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

gives $\chi(\mathcal{O}_C) = 1 - g(C) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C))$; the formula then follows from Riemann–Roch.

Remarks I.16

(i) Note that the genus of C is not the same as that of its normalization. More precisely, let $f : N \to C$ denote the normalization of C; we define a sheaf δ on C by the exact sequence

$$0 \to \mathcal{O}_C \to f_*\mathcal{O}_N \to \delta \to 0.$$

The cokernel δ is concentrated at the singular points of C, so that $H^1(C, \delta) = 0$, and $H^0(C, \delta) = \bigoplus_{x \in C} \delta_x$; $\delta = 0$ if and only if f is an isomorphism, that is C is smooth.

The associated cohomology exact sequence gives

$$g(C) = g(N) + \sum_{x \in C} \dim(\delta_x) .$$

Hence $g(C) > g(N)$ if C is singular; in particular, the condition $g(C) = 0$ implies that C is rational and smooth, that is $C \cong \mathbb{P}^1$.

(ii) The genus formula can also be written $2g-2 = \deg(K+C)_{|C}$ (see I.6). If C is smooth, we will now show that in fact $(\mathcal{O}_S(K+C))_{|C} = \omega_C$ (*adjunction formula*). For this, recall:

Fact I.17 *Let X and Y be two smooth varieties, and $j : X \hookrightarrow Y$ an embedding; write I for the ideal of X in Y. The sheaf $j^*I = I/I^2$ is then locally free of rank $\mathrm{codim}(X,Y)$ on X, and we have an exact sequence*

$$0 \to I/I^2 \xrightarrow{d} j^*\Omega^1_Y \xrightarrow{j^*} \Omega^1_X \to 0 \ .$$

Going back to the case $C \subset S$, we have $I = \mathcal{O}_S(-C)$, and this sequence becomes

$$0 \to \mathcal{O}_S(-C)_{|C} \to \Omega^1_{S|C} \to \omega_C \to 0 \ .$$

Considering the exterior powers gives the claimed equality.

(If C is singular, we still have $(\mathcal{O}_S(K+C))_{|C} = \omega_C$, where ω_C is the dualizing sheaf of C. But we will not use duality theory for singular curves.)

Historical Note I.18

The material of this chapter is the foundation of the theory of surfaces; it was all known before 1900. Linear systems, well understood on curves, were introduced in complete generality on sufaces by Max Noether ([N1]), and later studied very fully by Enriques ([E1]). The genus formula was proved by Noether in 1886 ([N2]), who used it to deduce the Riemann–Roch theorem, although assuming implicitly that $h^1(D) = h^1(\mathcal{O}_S) = 0$. The right form was given by Enriques in 1896 ([E1]), based on a result of Castelnuovo.

Noether's formula was proved in [N1]: Noether projects the surface birationally onto a singular surface in $\mathbb{P}^3$, and explicitly calculates the 3 invariants appearing in the formula; χ_{top} appears in the guise of the 'Zeuthen-Segre invariant'.

These geometers only considered effective curves; but the need to introduce 'virtual curves', that is, divisors, was quickly felt, especially by Severi. The theory is then complete. However, Serre's introduction of coherent sheaves in 1955 ([FAC]) revolutionized the presentation, turning most of the results into formalities. In 1956, Hirzebruch generalized the Riemann–Roch theorem to varieties of arbitrary dimension ([H]). His version contains Noether's formula; let us give a brief indication of his proof. A formal computation of characteristic classes gives

$p_1 = K^2 - 2\chi_{\text{top}}(S)$, where p_1 is the first Pontryagin class of S. Cobordism theory then shows that $p_1 = 3\tau$, where τ is the signature of the intersection form on $H^2(S, \mathbb{Z})$; indeed, the two sides of this equation are cobordism invariants, and they agree on $\mathbb{P}^2$. Finally, Hodge theory gives $\tau = 2 + 4h^2(\mathcal{O}_S) - b_2 = 4\chi(\mathcal{O}_S) - \chi_{\text{top}}(S)$. We deduce

$$K^2 + \chi_{\text{top}}(S) = 3\tau + 3\chi_{\text{top}}(S) = 12\chi(\mathcal{O}_S)\ .$$

Our presentation of the intersection form follows essentially [M1].