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to the memory of my mother,
Ljubica Kontić Djurdjević

Contents

Introduction

Geometric control theory provides the calculus of variations new perspectives that both unify its classic theory and outline new horizons toward which its theory extends. These perspectives grow from the theoretical foundations anchored in two important theorems not available to the classic theory of the calculus of variations.

The more immediate of these two theorems is the "maximum principle" of L. S. Pontryagin and his co-workers, obtained in the late 1950s. The maximum principle, a far-reaching generalization of Weierstrass's necessary conditions for strong minima, provides geometric conditions for a (strong) minimum of an integral criterion, called the "cost," over the trajectories of a differential control system. These conditions are based on the topological fact that an optimal solution must terminate on the boundary of the extended reachable set formed by the competing curves and their integral costs.

An important novelty of Pontryagin's approach to problems of optimal control consists of liberating the variations along the optimal curves from the constricting condition that they must terminate at the given boundary data. Instead, he considers variations that are infinitesimally near the terminal point and that generate a convex cone of directions locally tangent to the reachable set at the terminal point defined by the optimal trajectory. As a consequence of optimality, the direction of decreasing cost cannot be contained in the interior of this cone. This observation leads to the "separation theorem," which can be seen as a generalization of the classic Legendre transform in the calculus of variations, which ultimately produces the appropriate Hamiltonian function. The maximum principle states that the Hamiltonian that corresponds to the optimal trajectory must be maximal relative to the competing directions and that each optimal trajectory is the projection of an integral curve of the corresponding Hamiltonian vector field.

The maximum principle, in its original form, suffers from a serious limitation: It does not provide any information about the optimal trajectories whenever the reachable set formed by the competing curves is contained in a proper submanifold of ambient space, for then the conditions of the maximum principle are automatically satisfied by any competing curve, not because of optimality, but because the reachable set is contained in a manifold whose interior is void relative to the ambient manifold.

Geometric control theory forms a theoretical foundation for extensions of the maximum principle to optimal problems on arbitrary differentiable manifolds and circumvents the aforementioned limitation by providing the natural manifolds that contain the reachable sets and in which the reachable sets have a non-void interior. This theoretical foundation comprises important results concerning the topological and differential properties of the reachable sets and is an essential complement to modern optimal control theory.

The second important theorem, which has not been a part of the classic calculus of variations, but which is at the foundation of geometric control theory, is a theorem of W. L. Chow, published in 1939, concerning the reachable sets of integral curves for an arbitrary family of vector fields. Chow's theorem and the related Hermann-Nagano theorem concerning the existence of integral manifolds lead to a distinguished class of control systems, which I call Lie-determined, whose reachable sets admit easy descriptions in terms of Lie theoretical and algebraic criteria. This class is sufficiently large to include systems described by real analytic vector fields.

The qualitative theory of Lie-determined systems, further enriched with the maximum principle and the associated Hamiltonian formalism, provides a base for modern optimal control. This new subject removes some of the static confines of its classic predecessor, the calculus of variations, and offers a theory that is an exciting blend of differential equations, differential topology, geometry, and analysis. The diversity of problems that fit its theoretical framework accounts for the interdisciplinary character and makes the subject a crossroads for differential geometry, mechanics, and optimal control.

The subject matter is presented in two parts. The first part of this book deals with qualitative properties of the reachable sets defined by an arbitrary family of vector fields and establishes the theoretical foundation required for a study of optimal systems on differentiable manifolds. The second part of the book deals with optimality, and it progresses from linear quadratic optimal problems and time-optimal linear problems to more geometric problems on Lie groups and their homogeneous spaces.

In many respects this division of material parallels C. Carathéodory's treatment of the calculus of variations, published in two volumes in 1935. His first volume deals exclusively with the geometric properties of Pfaffian differential systems, and it was intended to provide the theoretical foundations for the optimal problems with differential constraints, known at the time as the problems of Lagrange. However, for all that those two volumes reflect profound insights into the mathematics of that period, the information contained in the first volume is never properly incorporated into the part on the calculus of variations. Nevertheless, Carathéodory's treatment of the subject, particularly his geometric point of view, considerably influenced the present exposition. In that spirit, I have included several problems treated by Carathéodory, which apart from the beauty of their solutions also illustrate the effectiveness of the new methods.

Among the contemporary books on the subject of the calculus of variations, P. Griffiths's book (1983) has had the strongest impact on this study. In particular, much of the material in this text concerning optimal problems on Lie groups owes its inspiration to Griffiths's book.

The choice of material presented in this book strongly reflects my own views that optimal control is also a natural setting for problems of geometry and mechanics, and therefore the corresponding theory should be a synthesis of the main ideas from all these subjects. For that reason, differential systems on Lie groups and their homogeneous spaces form a very important part of the text.

I have repeatedly defended the foregoing views by treating some of the most classic problems within the control-theory context. Partly for that reason, but mostly on its own merit, I include a derivation of the equations of motion for the "heavy top," a rigid body fixed at one of its points and subjected to the force of gravity. This derivation relies on the maximum principle for the correct Hamiltonian on the cotangent bundle of $SO_3(R)$. The classic equations of the heavy top are the integral curves of the corresponding Hamiltonian vector field expressed in the representation of the cotangent bundle of $SO_3(R)$ as the product $SO_3(R) \times \mathcal{L}^*$, with $\mathcal{L}^*$ equal to the dual of the Lie algebra of $SO_3(R)$.

The same formalism leads to a correct interpretation of a famous theorem of Kirchhoff, obtained in 1859, known as the "elastic kinetic analogue," which relates the equations describing the equilibrium configurations of a thin elastic bar to the equations describing the movements of the heavy top. The extension of this theorem to non-Euclidean spaces of constant curvature leads to new examples of integrable systems sharing the Kowalewski top relations among the coefficients corresponding to the principal moments of inertia.

Apart from looking at classic problems from new perspectives, this book also contains a detailed study of invariant optimal problems on Lie groups, including the use of symmetry based on the generalized Noether theorem and the connections with classic integrability theory.

The main ideas of optimal control are introduced through the quadratic regulator problem and the time-optimal problem, two of the most important problems of linear control theory. Each of these problems is presented in a self-contained manner, with particular emphasis on the phenomena that transcend the linear origins of these problems.

The quadratic regulator problem is treated within the most general class of linear quadratic problems that admit solutions. The resulting theory gives new proofs for the classic inequalities of Wirtinger and Hardy-Littlewood concerning the interpolation formulas for L_2 norms of functions whose derivatives also belong to L_2.

The solutions to the singular case lead to an optimal synthesis consisting of "jumps" and singular solutions. The synthesis is of "turnpike" type: The singular solutions are the turnpikes, and the jumps are its access routes. The geometric methods used to obtain these solutions are based on general geometric notions known as the Lie saturate, and they provide a theoretical base for solving problems with degenerate Legendre conditions. Such problems invariably lead to constrained Hamiltonian systems, and their solutions extend the pioneering work of P. A. M. Dirac in connection with the foundations of quantum mechanics. This class of problems further illustrates the geometric significance of Lie-determined systems.

Linear time-optimal problems, as well as the quadratic problem having bounds on controls, exhibit "chattering" and singular phenomena that also fall outside the classic theory, and so they provide important insights to further developments of the subject matter, while at the same time illustrating the distinctive contributions of optimal control to the classic calculus of variations.

This book is intended for students at the graduate level, although it probably contains more material than can reasonably be covered in one academic year. Some of the material concerning global controllability can be omitted during the first reading, without losing much continuity with the second part of the book on optimality.

My original plan was to write a self-contained presentation of this material accessible to a reader with a good undergraduate education in mathematics. Although I kept such a reader in mind, I soon realized that the paths to the most interesting aspects of the subject demanded of the reader more than I had originally expected. I chose to follow these paths to their natural ends, hoping that well-intentioned readers will still be able to get to the essence of the matter and feel inspired to master some of the mathematical details on their own.

Having understood the mathematical demands that the subject matter imposes, I have tried to make the presentation as accessible as possible by treating each topic in a self-contained manner. This concern accounts for the somewhat independent character of the chapters, particularly those dealing with optimal control. I hope that the pedagogical advantages of this style of presentation outweigh its mathematical inefficiencies.

Acknowledgments

A book exists in the imagination long before it exists in print. As early as 1980, Claude Lobry saw a need for a book on geometric control and challenged me to write it. The early chapters of this book, written much later, grew out of our discussions at the University of Bordeaux and the University of Nice. More important than the original inspiration, however, was the sustained insight and enthusiasm of Ivan Kupka. As a friend, mentor, and a colleague, his influence, which permeates this text all the way from practical details in the proofs to his imagined presence as the reader, has made this book better than it otherwise would have been.

The book was written in several stages spanning the period between two sabbatical leaves. I am grateful to Robbie Gardner and the members of the Department of Mathematics at the University of North Carolina at Chapel Hill for their hospitality and interest during the writing of the first part of the book.

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