Part one

Reachable sets and controllability

For the purposes of this book, a "control system" is any system of differential equations in which control functions appear as parameters. Our qualitative theory of control systems begins with the important geometric observation that each control determines a vector field, and therefore a control system can also be viewed as a family of vector fields parametrized by controls. A trajectory of such a system is a continuous curve made up of finitely many segments of integral curves of vector fields in the family.

This geometric view of control systems fits closely the theoretical framework of Sophus Lie for integration of differential equations and points to the non-commutativity of vector fields as a fundamental issue of control theory. The geometric context quickly reveals the Lie bracket as the basic theoretical tool, and the corresponding theory, known as geometric control theory, becomes a subject intimately connected with the structural properties of the enveloping Lie algebras and their integral manifolds. For this reason, our treatment of the subject begins with differentiable manifolds, rather than with $\mathbb{R}^n$ as is customary in the control-theory literature.

As natural a beginning as it may seem, particularly to the reader already familiar with differential geometry, this point of view is a departure from the usual presentation of control theory, which traditionally has been confined either to linear theory and the use of linear algebra or to control systems in $\mathbb{R}^n$, with an emphasis on optimality. The absence of geometric considerations and explicit mention of the Lie bracket in this literature can be attributed to the historical development of the subject. The following is a brief sketch of the main events:

Control theory, originally developed to satisfy the design needs of servomechanisms, under the name of "automatic control theory," became recognized as a mathematical subject in 1960, with the publication of the early papers of R. Kalman. Kalman challenged the accepted approach to control theory of that period, limited to the use of Laplace transforms and the frequency domain, by showing that the basic control problems could be studied effectively through the notion of a state of the system that evolves in time according to ordinary differential equations in which controls appear as parameters. Aside from drawing attention to the mathematical content of control problems, Kalman's work served as a catalyst for further growth of the subject. Liberated from the confines of the frequency domain and further inspired by the development of computers, automatic control theory became the subject matter of a new science called systems theory.

Systems theory grew out of a desire to merge automata theory, and artificial intelligence, and discrete and continuous control into a single subject concerned with input–output relations parametrized by the states of the system. The level of generality required to keep these subjects together was well beyond the realm of differential equations, and control systems quickly evolved into topological dynamical systems, or polysystems. Systems theory, itself a hybrid of control and automata theory, in its formative period looked to abstract dynamical systems and mathematical logic for its further growth. That initial orientation of systems theory, characteristic of the early 1960s, led away from geometric interpretations of linear theory and was partly responsible for the indifference with which R. Hermann's pioneering work of 1963 (relating Chow's theorem to control theory) was received by the mathematical community.

The significance of the Lie bracket for problems of control became clear around the year 1970 with publication of the papers of R. Brockett, H. Hermes, and C. Lobry, followed by the papers of P. Brunovsky, D. Elliott, A. Krener, H. J. Sussmann, and others. Thanks to that collective effort, differential geometry entered into an exciting partnership with control theory. The mid-1960s theorems of R. Hermann and T. Nagano concerning the existence of integral manifolds for singular distributions, and also the theorem of Chow, from 1939, found applications in problems of control. The control problem, on the other hand, through its distinctive concern for time-forward evolution of systems, led to its own theorems, marking the birth of geometric control theory. These theorems will compose the subject matter for the first part of this book.

1

Basic formalism and typical problems

Although differentiable manifolds have become familiar subjects to modern mathematicians, it still seems appropriate to begin with an introductory discussion of differentiable manifolds. Considering their importance in this exposition, the extra time required to examine their basic properties may well be worthwhile. This choice of introductory material, aside from making the text more accessible, will spare the reader the awkward task of having to translate the notations from other sources into the formalism used here.

1 Differentiable manifolds

Throughout this text, the state space M will be an n-dimensional real differentiable manifold. This means that M is a topological space such that at each point $p \in M$ there exists a neighborhood U of p and a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. It is assumed that n does not vary with the choice of a point p on M. The pair (ϕ, U) is called a *chart* at p, and U is called a coordinate neighborhood. We shall denote $\phi(p)$ by $(x_1(p), \ldots, x_n(p))$, where $x_1, \ldots, x_n$ are called coordinates of p. It follows that each coordinate x_i is a continuous function on U. For any charts (ϕ_1, U_1) and (ϕ_2, U_2) such that $U_1 \cap U_2 \neq \phi$, the restrictions of ϕ_1 and ϕ_2 to $U = U_1 \cap U_2$ are homeomorphisms onto open subsets $\phi_1(U)$ and $\phi_2(U)$ in $\mathbb{R}^n$. Then each of $\phi_2 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_2^{-1}$ is a mapping from an open set in $\mathbb{R}^n$ into $\mathbb{R}^n$. If we write $(y_1(x_1, \ldots, x_n), \ldots, y_n(x_1, \ldots, x_n))$ for the coordinates of such mappings, then the foregoing charts are said to be *compatible* if each of the functions $y_1(x_1, \ldots, x_n)$ is r times continuously differentiable in each of its arguments.

An atlas on M is a set of compatible charts on M that cover M. Two atlases on M are equivalent if their union is also an atlas on M. M and a class of equivalent atlases is called a C^r manifold. The particular equivalence class of atlases is called a *differentiable structure* on M. A set M may have more than one differentiable structure.

We shall admit only manifolds that satisfy each of the following additional conditions:

(a) For each pair of points p and q, there exist charts (ϕ_1, U_1) and (ϕ_2, U_2), such that $p \in U_1$, $q \in U_2$, and $U_1 \cap U_2 = \phi$. That is, points of M are separated by coordinate neighborhoods (M is Hausdorff).
(b) There exists a countable collection of compatible charts $\{(\phi_i, U_i)\}$ such that M is covered by the union of the sets $\{U_i\}$.

The combination of M together with all of these assumptions and its differentiable structure will be called a C^r *differentiable manifold*, or simply a *differentiable manifold*, with n being the dimension of M. For all practical purposes, we shall be interested in only C^∞ manifolds, which we shall refer to as *smooth*.

A very important class of manifolds in this development is composed of the manifolds that admit an analytic structure. A manifold M will be called analytic if for any compatible charts (ϕ_1, U_1) and (ϕ_2, U_2) the mapping $\phi_1 \circ \phi_2^{-1}$ is analytic as a mapping from an open set in $\mathbb{R}^n$ into $\mathbb{R}^n$. That is, each coordinate function $y_i(x_1, \ldots, x_n)$ is represented by its power-series expansion valid in a neighborhood of each point $(x_1, \ldots, x_n)$ in the domain of y_i.

In general, manifolds are obtained in the following way: Let $f_1(x_1, \ldots, x_n)$, $\ldots$, $f_m(x_1, \ldots, x_n)$ be any C^r $(r \geq 1)$ functions defined on an open set Ω in $\mathbb{R}^n$. Assume that the Jacobian matrix $(\partial f_i / \partial x_j)$ has constant rank k at each point $x = (x_1, \ldots, x_n)$ in Ω. We may assume, without any loss of generality, that the matrix $(\partial f_i / \partial x_j)$, where $1 \leq i \leq k$, $1 \leq j \leq k$, is nonsingular. Now consider the transformation ψ defined by

$$\begin{aligned} y_i &= f_i(x_1, \ldots, x_n), \qquad 1 \leq i \leq k, \\ y_i &= x_i, \qquad i > k. \end{aligned} \tag{1}$$

Because the Jacobian of the foregoing mapping is nonsingular at each $x \in \Omega$, it follows, by the inverse-mapping theorem, that at each $x \in \Omega$, ψ has an inverse ϕ valid in some neighborhood U. So each coordinate x_i is a function of $y_1, \ldots, y_n$. In terms of the new coordinates $y_1, \ldots, y_n$,

(a) $f_i(y_1, \ldots, y_n) = y_i$, $1 \leq i \leq k$, and
(b) $(\partial f_i / \partial y_j)(y_1, \ldots, y_n) = 0$ for i, j such that $i > k$, $j > k$.

Problem 1 Prove condition (b).

Cambridge University Press
978-0-521-49502-8 - Geometric Control Theory
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It follows from (b) that each function f_i, for $i > k$, is constant as a function of the coordinates $y_{k+1}, \dots, y_n$. This argument shows that $y_{k+1}, \dots, y_n$ can be used as coordinates for each point x in the set $M = \{x : f_1(x) = \text{const}, f_2(x) = \text{const}, \dots, f_m(x) = \text{const}\}$ topologized by the relative topology from $\mathbb{R}^n$. It then follows that each connected component of M is an $(n-k)$-dimensional differentiable manifold, with the system of charts as described earlier. We shall refer to this fact as the "constant-rank theorem."

Example 1 Each n-dimensional real vector space V becomes an n-dimensional smooth (or analytic) manifold under the correspondence $\phi(p) = (x_1, \dots, x_n)$, with $(x_1, \dots, x_n)$ the coordinates of p relative to a fixed choice of basis $v_1, \dots, v_n$ in V. (V, ϕ) is a global chart for V. Any other chart (V, ϕ) is compatible as long as $\psi \circ \phi^{-1}$ is a smooth (or analytic) mapping on $\mathbb{R}^n$. A function f on V is smooth (or analytic) if $f(x_1, \dots, x_n)$ is a smooth (or analytic) function on an open subset of $\mathbb{R}^n$.

Example 2 Any open subset of a manifold M is a manifold with its coordinate charts and the differentiable structure inherited from M.

Example 3 Let $\mathrm{GL}(V)$ denote the set of all nonsingular linear transformations of an n-dimensional real vector space V. Then $\mathrm{GL}(V)$ is an n^2-dimensional real analytic manifold, for the following reasons:

Use $\mathrm{end}(V)$ to denote the vector space of all linear endomorphisms on V. It follows that $\mathrm{end}(V)$ is an n^2-dimensional vector space, and by Example 1 is a real analytic manifold of the same dimension. Because the determinant of an $n \times n$ matrix is a continuous function of the entries of the matrix, $\mathrm{GL}(V)$ is an open subset of $\mathrm{end}(V)$. By Example 2, it inherits the manifold structure of $\mathrm{end}(V)$. $\mathrm{GL}(V)$ consists of two connected components $\mathrm{GL}^+(V)$ and $\mathrm{GL}^-(V)$. $\mathrm{GL}^+(V)$ is the component that contains the identity.

Example 4 The n-dimensional sphere S^n can be realized as the level surface of $f(x_1, \dots, x_{n+1}) = x_1^2 + \dots + x_{n+1}^2 = 1$; f satisfies the constant-rank condition at each point $p = (x_1, \dots, x_{n+1})$ such that $\sum_{i=1}^{n+1} x_i^2 \neq 0$. On S^n, $\sum_{i=1}^{n+1} x_i^2 = 1$, and at least one of the components x_i is not zero. Hence the remaining components serve as the coordinates on $\mathbb{R}^n$. For instance, the open hemisphere U on S^n consisting of all points $p = (x_1, \dots, x_{n+1})$, with $x_{n+1} > 0$, is the graph of the mapping $x_{n+1} = \sqrt{1 - (x_1^2 + \dots + x_n^2)}$ defined on the open disk $x_1^2 + \dots + x_n^2 < 1$ in $\mathbb{R}^n$.

The stereographic projection of any point p from $p_0 = e_{n+1}$ onto the equatorial plane $x_{n+1} = 0$ is given by $y_i = x_i/(1 - x_{n+1})$, $i = 1, \ldots, n$, where $y_1, \ldots, y_n$ is another system of coordinates on $V = S^n - \{p_0\}$. These two charts are compatible in $U \cap V$ because

$$y_i(x_1, \ldots, x_n) = \frac{1}{1 - \sqrt{1 - (x_1^2 + \cdots + x_n^2)}}, \qquad i = 1, \ldots, n,$$

is a smooth mapping from the open annulus $0 < x_1^2 + \cdots + x_n^2 < 1$ onto the open set $y_1^2 + \cdots + y_n^2 > 1$. The inverse map is given by

$$x_i = \frac{2}{1 + y_1^2 + \cdots + y_n^2} y_i, \qquad i = 1, \ldots, n.$$

The sphere S^n cannot be covered by a single coordinate chart because of its compactness (as is well known in cartography) (Figure 1.1).

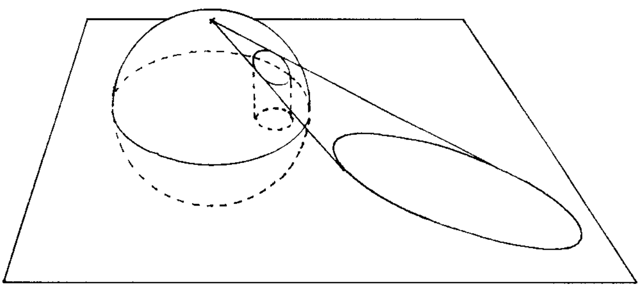

Fig. 1.1.

1.1 Differentiable mappings

Let M be a differentiable manifold, W an open subset of M, and $f : W \to \mathbb{R}^1$. Then f is said to be *differentiable* if for any local chart (ϕ, U) on M, $f \circ \phi^{-1}$ is differentiable as a function from the open subset $\phi(U \cap W)$ of $\mathbb{R}^n$ into $\mathbb{R}^1$.

A continuous mapping $F : M_1 \to M_2$, with both M_1 and M_2 differentiable manifolds, is said to be *differentiable* if for any function f differentiable on M_2, $f \circ F$ is a differentiable function on M_1. We shall write F^*f for the function $f \circ F$, and we shall refer to F^*f as the pull-back of f under F.

Let F be a differentiable mapping from M_1 to M_2, with $\dim M_1 = m$ and $\dim M_2 = n$. Let (ϕ, U) be a chart at p_0 on M_1, and (ψ, V) a chart at $F(p_0)$

on M_2. Write $\phi(p) = (x_1, \ldots, x_m)$ for any $p \in U$, and $\psi(q) = (y_1, \ldots, y_n)$ for any $q \in V$. Then each coordinate function y_i is a differentiable function on M_2. Hence the pull-back F^*y_i is a differentiable function on M_1. But then $F^*y_i \circ \phi^{-1}$ is a differentiable function on an open set in $\mathbb{R}^m$. Therefore, the mapping

$$\psi \circ F \circ \phi^{-1} \text{ is differentiable as a mapping from } \quad \mathbb{R}^m \quad \text{into} \quad \mathbb{R}^n. \qquad (2)$$

Conversely, if F is any continuous mapping from M_1 into M_2 such that F satisfies (2) for any charts (ϕ, U) on M_1 and (ψ, V) on M_2, then F is differentiable.

In particular, any curve $\sigma : I \to M$ is differentiable if $\sigma^* f$ is a differentiable function on the interval I for all differentiable functions f on M. It then follows from (2) that σ is a differentiable curve if and only if $\phi \circ \sigma : I \to \mathbb{R}^n$ is differentiable for any chart (ϕ, U) on M.

1.2 The tangent space

Each point p on a differentiable manifold M defines a real vector space T_pM of the same dimension as M called the tangent space of M at p.

If M were a subset of $\mathbb{R}^n$ such that differentiable curves on M agreed with the restrictions of differentiable curves in $\mathbb{R}^n$ to M, then T_pM could be defined as follows: $v \in T_pM$ if and only if there exists a differentiable curve σ with values in M defined on an interval $(-\varepsilon, \varepsilon)$ such that $\sigma(0) = p$ and such that $(d\sigma/dt)|_{t=0} = v$. However, such a definition would not be intrinsic, for it would depend on the ambient space $\mathbb{R}^n$ as well as M. The intrinsic definition of the tangent space is an abstraction of the preceding idea that does not make any reference to the ambient space, and it goes as follows:

Let $C(p)$ be the space of all differentiable curves σ on M defined on open intervals $(-\varepsilon, \varepsilon)$ in $\mathbb{R}^1$ that satisfy $\sigma(0) = p$. Curves α and β in $C(p)$ are said to be equivalent if $(d/dt)\phi \circ \alpha(t)|_{t=0} = (d/dt)\phi \circ \beta(t)|_{t=0}$ for any chart (ϕ, U) at p. It follows that such equivalence is well defined; that is, if (ϕ, U) and (ψ, V) are two charts at p, then

$$\frac{d}{dt}\phi \circ \alpha(t)|_{t=0} = \frac{d}{dt}\phi \circ \beta(t)|_{t=0}$$

if and only if

$$\frac{d}{dt}\psi \circ \alpha(t)|_{t=0} = \frac{d}{dt}\psi \circ \beta(t)|_{t=0}.$$

Problem 2 Prove the preceding statement.

T_pM is defined as the set of all equivalence classes of $C(p)$. If $\alpha \in C(p)$, let $[\alpha]$ denote its equivalence class. For any chart (ϕ, U) at p, the mapping $\bar{\phi} : T_pM \to \mathbb{R}^n$ defined by $\bar{\phi}([\alpha]) = (d/dt)\phi \circ \alpha(t)|_{t=0}$ is one-to-one and onto $\mathbb{R}^n$. In fact, for any v in $\mathbb{R}^n$, $\alpha(t) = \phi^{-1} \circ (\phi(p) + tv)$ is a curve such that $\bar{\phi}([\alpha]) = v$. We define the linear structure on T_pM so that $\bar{\phi}$ becomes an isomorphism. It remains to show that the linear structure on T_pM is independent of the particular chart.

Now let (ψ, V) be another chart. Let $\bar{\phi}([\alpha]) = v$, and let $\bar{\psi}([\alpha]) = w$. It follows that $w = (d/dt)\psi \circ \alpha(t)|_{t=0} = (d/dt)\psi \circ \phi^{-1} \circ \phi \circ \alpha(t)|_{t=0}$. Therefore the coordinates of v and w transform according to the following formula:

$$w_i = \sum_{j=1}^{n} \frac{\partial y_i}{\partial x_j} v_j,$$

where $y_i(x_1, \ldots, x_n)$, $i = 1, 2, \ldots, n$, denotes the coordinates of the mapping $\psi \circ \phi^{-1}$. Hence the structure of T_pM is canonical in the sense that it is independent of the choice of coordinates.

1.3 The cotangent space

The cotangent space of M at p, denoted by T_p^*M, will be defined in terms of the differentiable functions f, defined in some neighborhood of p, that satisfy $f(p) = 0$. Let $F(p)$ denote such a class of functions. $F(p)$ will have a natural vector-space structure provided that functions that agree on a common domain are regarded as equal. (The domains of elements of $F(p)$ need not all be the same.)

For any f and g in $F(p)$, we shall say that f and g are equivalent if $d(f \circ \phi^{-1})|_{x=\phi(p)} = d(g \circ \phi^{-1})|_{x=\phi(p)}$ for any local chart (ϕ, V) at p. We shall write,

$$f \circ \phi^{-1}(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) \quad \text{and} \quad d(f \circ \phi^{-1}) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

It follows that if $f \sim \bar{f}$, and if $g \sim \bar{g}$, then $\alpha f \sim \alpha \bar{f}$ and $\alpha g \sim \alpha \bar{g}$ for any real number α, and $f + g \sim \bar{f} + \bar{g}$.

Problem 3 Prove the preceding statements.

Therefore, the space of equivalence classes becomes a vector space with the operations

$$\alpha[f] + \beta[g] = [\alpha f + \beta g].$$

The cotangent space T_p^*M is equal to the foregoing vector space of equivalence classes.

For each chart (ϕ, U), the mapping $\bar{\phi} : F(p)/_\sim \to \mathbb{R}^n$ defined by $\bar{\phi}([f]) = d(f \circ \phi^{-1})$ is a linear isomorphism. If $x_i(q)$ is the ith coordinate function of ϕ, then $f_i(q) = x_i(q) - x_i(p)$, $q \in U$, is an element of $F(p)$, and $\bar{\phi}([f_i]) = e_i$. So $[f_1], \ldots, [f_n]$ is a basis for $F(p)/_\sim$. Another chart (ψ, V) produces its own basis $[g_1], \ldots, [g_n]$, with $g_i(q) = y_i(q) - y_i(p)$, $\quad q \in V$. Let $[f]$ be an arbitrary element of T_p^*M. Then $[f] = \sum_{i=1}^n v_i[f_i] = \sum_{i=1}^n w_i[g_i]$. It then follows that the coordinates $(w_1, \ldots, w_n)$ are related to the coordinates $(v_1, \ldots, v_n)$ via the following formula:

$$v_i = \sum_{j=1}^n \frac{\partial y_j}{\partial x_i} w_j.$$

We end this section by showing the duality between the elements of T_p^*M and those of T_pM. For any $f \in F(p)$ and any $\sigma \in C(p)$, consider the pairing

$$\langle [f], [\sigma] \rangle = \frac{d}{dt} f \circ \sigma |_{t=0}.$$

Because $f \circ \sigma = f \circ \phi^{-1} \circ \phi \circ \sigma$, it follows that the foregoing pairing is well defined and is bilinear. More explicitly,

$$\langle [f], [\sigma] \rangle = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial \sigma_i}{dt},$$

with $d(f \circ \phi^{-1}) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$, and with $(d/dt)\phi \circ \sigma(t)|_{t=0} = (d\sigma_1/dt, \ldots, d\sigma_n/dt)$. Therefore each element of T_p^*M is a linear functional on T_pM, and hence

$$T_p^*M = (T_pM)^*.$$

The preceding definitions draw attention to an important distinction between tangent spaces and cotangent spaces. Let M and N be differentiable manifolds, and $\Phi : M \to N$ a differentiable map. We have already mentioned that Φ *pulls back* differentiable functions of N into the differentiable functions on M. However, for differentiable curves the situation is different. For any differentiable curve σ on M, $\Phi \circ \sigma$ is a differentiable curve on N. Thus Φ *pushes forward* curves on M into curves on N. We shall write $\Phi_*\sigma$ for the curve $\Phi \circ \sigma$.

Both the push-forward Φ_* and the pull-back Φ^* induce linear mappings called the tangent map of Φ and the differential of Φ. The tangent map of Φ will be denoted by $T\Phi$ and is a linear mapping from T_pM into $T_{\Phi(p)}N$, whereas the differential $d\Phi$ of Φ is a linear mapping from $T^*_{\Phi(p)}M$ into T^*_pM. These mappings are defined by the following formulas:

$$T\Phi[\alpha] = [\Phi_*\alpha] \quad \text{for any } [\alpha] \text{ in } T_pM, \tag{3a}$$

$$d\Phi[f] = [\Phi^* f] \quad \text{for any } [f] \text{ in } T^*_{\Phi(p)}N. \tag{3b}$$

Problem 4 Show that the mappings in (3) are well defined and that they are both linear.

In terms of the local charts (ϕ, U) at p in M and (ψ, V) at $\Phi(p)$ in N, the preceding formulas translate as follows. Let

$$v = \frac{d}{dt}\phi \circ \alpha(t)|_{t=0}, \qquad w = \frac{d}{dt}\psi \circ \Phi \circ \alpha|_{t=0},$$

and

$$\Phi_i(x_1, \ldots, x_n) = \psi_i \circ \Phi \circ \phi^{-1}(x_1, \ldots, x_n), \qquad i = 1, 2, \ldots, m.$$

Then (3a) is equivalent to

$$w_i = \sum_{j=1}^{n} \frac{\partial \Phi_i}{\partial x_j} v_j, \qquad i = 1, 2, \ldots, m. \tag{4}$$

In order to get an analogous expression for (3b), let f be a differentiable function on N at $\Phi(p)$, and g its pull-back $\Phi^* f$. Denote $g(x_1, \ldots, x_n) = \Phi^* f \circ \phi^{-1}(x_1, \ldots, x_n)$, and $f(y_1, \ldots, y_m) = f \circ \psi^{-1}(y_1, \ldots, y_m)$. Then $g(x_1, \ldots, x_n) = f(\Phi_1(x), \ldots, \Phi_m(x))$, and hence

$$\frac{\partial g}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial \Phi_j}{\partial x_i} \frac{\partial f}{\partial y_j} \quad \text{for each } i = 1, 2, \ldots, n. \tag{5}$$

Equation (4) says that the tangent vectors transform according to the Jacobian matrix $(\partial\Phi_i/\partial x_j)$, and (5) shows that the covectors transform according to its transpose. These transformations are often called the contravariant and covariant, respectively.