Cambridge University Press
978-0-521-48573-9 - A Primer of Nonlinear Analysis
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Preliminaries and notation

This chapter contains the notation and some preliminary tools used throughout the book. Almost always, the results are not quoted in the most general form, but in a way appropriate to our purposes; nevertheless some of them are actually slightly more general than we strictly need. For more detail we refer to any book of (linear) Functional Analysis (for example [**Y**] or [**Br**] for topics reviewed in sections 0.1–0.4; to [**Br**], [**KFS**], [**GT**] for 0.5–0.6).

0.1 Some notation and definitions

$\mathbb{R}^n$ will denote the n-dimensional Euclidian space with scalar product $x \cdot y$ and norm given by $|x|^2 = x \cdot x$.

$X, Y, Z, \ldots$ denote (real) Banach spaces with norm $\|.\|_X$, $\|.\|_Y$, etc., respectively (the subscript will be omitted if no possible confusion arises). $B(x^*, r)$ denotes the ball $\{x \in X : \|x - x^*\| < r\}$ and $B(r)$ stands for $B(0, r)$.

If X^* is the topological dual of X the symbol $\langle .,. \rangle$ will indicate the duality pairing between X and X^*.

Let $\{x_n\}$ be a sequence in X. We say that x_n converges (strongly) to $x \in X$, written as $x_n \to x$, if $\|x_n - x\| \to 0$ as $n \to \infty$; we say that x_n converges weakly to x, written as $x_n \rightharpoonup x$, if $\langle \psi, x_n - x \rangle \to 0$ as $n \to \infty$ for all $\psi \in X^*$.

Let X be a Banach space and let V be a closed subspace of X. A *topological complement* of V in X is a *closed* subspace W of X such that $V \cap W = \{0\}$ and $X = V \oplus W$; $V \oplus W$ is called a *splitting* of X.

Recall also that, associated with such a splitting, there are (continuous) projections P and Q onto V and W respectively.

0.2 Continuous mappings

We will deal with continuous maps $f : U \to Y$, where U is an open subset of X. Continuity means that $f(x_n) \to f(x)$ (strongly) for any sequence x_n strongly convergent to $x \in X$. The set of all continuous $f : U \to Y$ will be denoted by $C(U, Y)$.

0.3 Integration

For continuous maps $f : [a, b] \to Y$ the definition of the Cauchy integral is given as in the elementary case, as the (strong) limit of the finite sums $\Sigma f(\xi_i)(t_i - t_{i-1})$ (with obvious meaning).

From

$$\|\Sigma_i f(\xi_i)(t_i - t_{i-1})\| \le \Sigma_i \|f(\xi_i)(t_i - t_{i-1})\| \le \Sigma_i \|f(\xi_i)\|(t_i - t_{i-1})$$

there follows immediately the inequality

$$\left\| \int_a^b f(t)\mathrm{d}t \right\| \le \int_a^b \|f(t)\|\mathrm{d}t.$$

0.4 Linear continuous maps

The space of linear continuous maps $A : X \to Y$ will be denoted by $L(X, Y)$. The range of A, $R(A)$, is the linear space $\{A(x) : x \in X\}$. Sometimes, when $Y = X$, we will use the notation $L(X)$ instead of $L(X, X)$. Equipped with the norm

$$\|A\| = \sup\{\|A(x)\| : \|x\| \le 1\},$$

$L(X, Y)$ is a Banach space. The identity map in $L(X)$ will be denoted by I_X.

Hereafter, for linear maps, the notation Ax or $A[x]$ may replace $A(x)$.

An *eigenvalue* of $A \in L(X)$ is a $\mu \in \mathbb{C}$ such that the equation $Ax = \mu x$ has solutions $x \ne 0$. Any such solution is an *eigenvector* associated to μ and Ker $(\mu I - A)$ is the *eigenspace* associated to μ. We will be mainly interested in the case when $A \in L(X)$ is compact, namely when A is completely continuous, if this is the case, the following results hold.

Theorem 0.1 **(Fredholm Alternative)** *Let $A \in L(X)$ be compact and $\mu \ne 0$. Then*

(i) Ker $(\mu I - A) = \{0\}$ *if and only if* $R(\mu I - A) = X$,

(ii) *$R(\mu I - A) = [\text{Ker } (\mu I - A^*)]^\perp = \{u \in X : \langle\psi, u\rangle = 0$ for all $\psi \in$ Ker $(\mu I - A^*)\}$.*

Moreover one has the following

Theorem 0.2 *Let $A \in L(X)$ be compact and $\mu \neq 0$. Then*

(i) *Ker $(\mu I - A)$ is finite-dimensional and Range$(\mu I - A)$ is closed,*

(ii) *the sequence Ker $((\mu I - A)^n)$ $(n \in \mathbb{N})$ is increasing, that is Ker $((\mu I - A)^m) \subset$ Ker $((\mu I - A)^{m+1})$ for all $m \geq 1$,*

(iii) *there exists a finite $p \in \mathbb{N}$ such that Ker $((\mu I - A)^p)$ =Ker $((\mu I - A)^q)$ if and only if $q \geq p$.*

The (algebraic) multiplicity of μ is the dimension of the linear subspace

$$\cup_{n\in\mathbb{N}}\text{Ker}((\mu I - A)^n) = \text{Ker}((\mu I - A)^p).$$

It is worth pointing out that the *algebraic* multiplicity of μ is, in general, different from the *geometric* multiplicity, defined as the dimension of Ker $(\mu I - A)$ (algebraic and geometric multiplicity coincide for self-adjoint operators on Hilbert spaces). Hereafter, by the multiplicity of an eigenvalue $\mu \neq 0$ of a completely continuous $A \in L(X)$ we will always mean the algebraic multiplicity. An eigenvalue will be said to be *simple* if its multiplicity is 1.

0.5 Function spaces

Let Ω be an open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ and closure $\overline{\Omega}$.
We will use standard notation for spaces of continuous or differentiable real-valued functions $C^k(\overline{\Omega})$ $(k \geq 0)$, for Lebesgue spaces $L^p(\Omega)$ $(1 \leq p < \infty)$ or $L^\infty(\Omega)$. In some cases we will write $C(\overline{\Omega})$ instead of $C^0(\overline{\Omega})$. The spaces above are Banach spaces under the norms defined, respectively, by

$$\|u\|_C = \sup\ \{|u(x)| : x \in \overline{\Omega}\},$$

$$\|u\|_{C^k} = \sum_{0\leq|\beta|\leq k} \|D^\beta u\|_C \quad (\beta \text{ is a multi index}),$$

$$\|u\|_{L^p} = \left[\int_\Omega |u|^p\right]^{1/p}$$

(the symbol dx will be omitted whenever there is no ambiguity)

$$\|u\|_{L^\infty} = \text{ess sup}\ \{|u(x)| : x \in \Omega\}.$$

For $k \geq 0$ and $0 < \alpha \leq 1$, $C^{k,\alpha}(\overline{\Omega})$ denotes the space of Hölder functions with exponent α, namely the $u \in C^k(\overline{\Omega})$ such that, for all multi-index $\beta, |\beta| = k$,

$$\sup\left\{\frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha} : x, y \in \Omega, x \neq y\right\} < \infty.$$

For $k = 0$ and $\alpha = 1$, $C^{0,1}(\overline{\Omega})$ is nothing but the space of Lipschitz-continuous functions on $\overline{\Omega}$.
Equipped with the norm

$$\|u\|_{C^{k,\alpha}} = \|u\|_{C^k} + \\ + \sup\left\{\frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha} : x, y, \in \Omega, x \neq y,\ |\beta| = k\right\},$$

$C^{k,\alpha}(\overline{\Omega})$ is a Banach space.
In some cases, we shall also work with Sobolev spaces $H^{k,p}(\Omega)$ ($k \geq 1, p \in [1, \infty)$) equipped with the norm

$$\|u\|_{H^{k,p}} = \sum_{0 \leq |\beta| \leq k} \|D^\beta u\|_{L^p}.$$

The notation H^k will stand for $H^{k,2}$ while $H_0^k(\Omega)$ will denote the closure of $C_0^\infty(\Omega)$, the space of C^∞ functions with compact support in Ω, under the norm $\|u\|_{H^{k,2}}$. Among others, let us recall the following result.

Theorem 0.3 **(Poincaré Inequality)** *Let Ω be bounded. Then there exists a constant $c = c(\Omega)$ such that*

$$\int_\Omega |u|^2 \leq c \int_\Omega |\nabla u|^2 \text{ for all } u \in H_0^1(\Omega).$$

As a consequence, $\|\nabla u\|_{L^2}$ is a norm in $H_0^1(\Omega)$ equivalent to $\|u\|_{H^{1,2}}$.

In addition to the Poincaré Inequality one has that the *embedding* of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact (Rellich's Theorem). Let us recall that X is embedded in Y, $X \hookrightarrow Y$, if $X \subset Y$ and the inclusion $\imath : X \to Y$ is continuous. If $X \hookrightarrow Y$ then $\exists\, c > 0$ such that

$$\|u\|_Y \leq c\|u\|_X, \text{ for all } u \in X.$$

If the inclusion $\imath : X \to Y$ is *compact* we will write $X \hookrightarrow\hookrightarrow Y$.

The following result is a particular case of the "Sobolev Embedding Theorems".

Theorem 0.4 *Suppose that Ω is bounded open set in $\mathbb{R}^n$, with boundary $\partial\Omega$ of class $C^{0,1}$, and let $k \geq 1$ and $1 \leq p \leq \infty$.*

(i) *If $kp < n$, then $H^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, for all $1 \leq q \leq np/(n - kp)$.*
(ii) *If $kp = n$, then $H^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, for all $q \in [1, \infty)$.*

(iii) *If* $kp > n$, *then* $H^{k,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$, *where* $\alpha = k - n/p$ *if* $k - n/p < 1$; $\alpha \in [0,1)$ *is arbitrary if* $k - n/p = 1$ *and* $p > 1$; $\alpha = 1$ *if* $k - n/p > 1$.

In addition, there result the following.

(i′) *If* $kp < n$, *then* $H^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, *for all* $1 \le q < np/(n-kp)$.
(ii′) *If* $kp = n$, *then* $H^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, *for all* $q \in [1,\infty)$.
(iii′) *If* $kp > n$, *then* $H^{k,p}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$.

0.6 Elliptic boundary value problems

Let Ω be a bounded domain (i.e. open connected) in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ (this will always be understood hereafter) and let $\mathcal{L}$ denote the differential operator

$$\mathcal{L} = \sum_{1 \le i,j \le n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) \tag{0.1}$$

where

$$a_{ij} = a_{ji} \in C^\infty(\overline{\Omega}). \tag{0.2}$$

$\mathcal{L}$ is (uniformly) elliptic if there exists $\alpha > 0$ such that

$$\sum_{1 \le i,j \le n} a_{ij}(x)\xi_i\xi_j \ge \alpha|\xi|^2, \text{ for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n \tag{0.3}$$

Throughout the book, any elliptic operator will be an *elliptic operator with smooth coefficients*, namely an $\mathcal{L}$ of the form (0.1) and such that (0.2)–(0.3) hold.

Consider the Dirichlet Boundary Value Problem (b.v.p. for short)

$$\left.\begin{aligned} -\mathcal{L}u &= h(x) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}\right\} \tag{0.4}$$

where h is given a function on Ω.
Let $h \in L^2(\Omega)$; a *weak solution* of (0.4) is a $u \in H_0^1(\Omega)$ such that

$$\sum_{1 \le i,j \le n} \int_\Omega a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} = \int_\Omega hv, \text{ for all } v \in C_0^\infty(\Omega).$$

If u is a weak solution of (0.4) and $u \in C^2(\Omega)$, then u is a classical solution.

Theorem 0.5 *Suppose* $\mathcal{L}$ *is an elliptic operator. Then the following results hold.*

(i) *Let* $h \in L^p(\Omega)$, $2 \le p < \infty$. *Then* (0.4) *has a unique (weak) solution* $u \in H_0^1(\Omega) \cap H^{2,p}(\Omega)$ *and the following estimate holds:*

$$\|u\|_{H^{2,p}} \le c\|h\|_{L^p}.$$

(ii) *If* $h \in L^\infty(\Omega)$ *then* $u \in C^{1,\alpha}(\overline{\Omega})$ *for any* $0 < \alpha < 1$ *and*
$$\|u\|_{C^{1,\alpha}} \le c\,\|h\|_{L^\infty}.$$
(iii) *If* $h \in C^{0,\alpha}(\overline{\Omega})$ *then* $u \in C^{2,\alpha}(\overline{\Omega})$ *is a classical solution of* (0.4) *and*
$$\|u\|_{C^{2,\alpha}} \le c\,\|h\|_{C^{0,\alpha}}.$$
In the above c *stands for a positive constant, depending on* Ω.

As a consequence of the preceding results, we can define an operator $K : L^2(\Omega) \to L^2(\Omega)$ (the *Green* operator of $-\mathcal{L}$ with zero Dirichlet boundary conditions) setting $Ku = v$ if and only if $-\mathcal{L}v = u$, $v \in H_0^1(\Omega)$. From the Rellich Theorem it follows immediately that K is compact.

Of course, the Green operator K might also be defined between other function spaces. For example, one could consider K as a map for $C(\overline{\Omega})$ into itself. Using Theorem 0.5 (ii) jointly with the Ascoli Theorem one still deduces that K is compact.

Given a function $m \in L^\infty(\Omega)$, let us consider the linear eigenvalue problem

$$\left.\begin{aligned} -\mathcal{L}u &= \lambda m u \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{0.5}$$

An eigenvalue of (0.5) is a λ such that (0.5) has a solution $u \neq 0$. Any $\phi \neq 0$ satisfying (0.5) is an eigenfunction associated to the eigenvalue λ. If we set $\mu = 1/\lambda$ and $K_m(u) = K(mu)$, problem (0.5) is equivalent to $\mu u = K_m u$. The eigenvalues λ_k of (0.5) correspond, through $\mu_k = 1/\lambda_k$ to the eigenvalues of K_m. The multiplicity of λ_k is the multiplicity of μ_k. In some cases we will write $\lambda_k(m)$ or $\lambda_k(\Omega)$ to highlight the dependence of the eigenvalues of (0.5) on m or Ω.

Theorem 0.6 *Let* $m \in L^\infty(\Omega)$, $m \ge 0$ *and* $m(x) > 0$ *in a set of positive measure.*

(i) *Problem* (0.5) *has a sequence*
$$0 < \lambda_1(m) < \lambda_2(m) \le \ldots \le \lambda_k(m) \le \ldots$$
of eigenvalues such that $\lambda_k(m) \to +\infty$ *as* $k \to \infty$. *The first eigenvalue* $\lambda_1(m)$ *is simple and the corresponding eigenfunctions do not change sign in* Ω. *We will let denote* ϕ_1, *(sometimes only* ϕ*) the eigenfunction such that* (a) $\phi > 0$ *in* Ω *and* (b) $\int_\Omega \phi^2 = 1$.
We will also let ϕ_k *denote the eigenfunctions corresponding to* λ_k *normalized by*
$$\int_\Omega \phi_h \phi_k = \delta_{hk} = \begin{cases} 1 & \text{if } h = k, \\ 0 & \text{if } h \neq k. \end{cases}$$
When $m \equiv 1$ *we will simply write* λ_k *instead of* $\lambda_k(1)$.

(ii) (Comparison property) *If* $m \le M$ *in* Ω *then* $\lambda_k(m) \ge \lambda_k(M)$; *if* $m < M$ *in a subset of positive measure then* $\lambda_k(m) > \lambda_k(M)$. *In particular, if* $m < \lambda_k$ *(resp.*$> \lambda_k$*) then* $\lambda_k(m) > 1$ *(resp.*< 1*).*

(iii) (Variational characterization) *There results*

$$\lambda_k^{-1}(m) = \max\left\{ \int_\Omega mv^2 : v \in H_0^1(\Omega), \quad \int_\Omega \sum a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} = 1 \right., \\ \left. \int_\Omega v\phi_i = 0, \text{ for all } i = 1, \dots, k-1 \right\}.$$

(iv) (Continuity property) $\lambda_1(m)$ *depends continuously on* m *in the* $L^{n/2}(\Omega)$ *topology.*

(v) *Let* Ω' *be a bounded domain, such that* $\Omega' \subset \Omega$. *Then* $\lambda_k(\Omega') \ge \lambda_k(\Omega)$ *for all* $k \ge 1$.

Consider the non-homogenous b.v.p.

$$\left.\begin{aligned} -\mathcal{L}u &= \lambda mu + h \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}\right\} \tag{0.6}$$

with, say, $h \in L^2(\Omega)$.

From the Fredholm Alternative Theorem 0.1 we get the following.

Theorem 0.7

(i) *If* λ *is not an eigenvalue of* (0.5), *then* (0.6) *has a unique solution for all* $h \in L^2(\Omega)$;

(ii) *if* λ *is an eigenvalue of* (0.5), *then* (0.6) *has a solution if and only if* $\int_\Omega h\phi_k = 0$ *for any* k *such that* $\lambda = \lambda_k$.

According to Theorem 0.4 (iii) all the preceding discussion can be carried over taking $X = C^{2,\alpha}(\overline{\Omega})$, $h \in C^{0,\alpha}(\overline{\Omega})$ and m smooth.

The arguments above apply to Sturm–Liouville Problems

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha \frac{\mathrm{d}}{\mathrm{d}x} u\right) + \beta u = h(x) \quad (0 < x < \pi), \\ a_0 u(0) + b_0 u'(0) = a_1 u(\pi) + b_1 u'(\pi) = 0, \end{aligned}\right\} \tag{0.7}$$

where $\alpha \in C^1([0,\pi])$, $\beta \in C((0,\pi])$, $\alpha, \beta > 0$ on $[0,\pi]$, and a_0, b_0, a_1, b_1 are such that $(a_0^2 + b_0^2)(a_1^2 + b_1^2) \ne 0$.

In fact, it is known [**D1**] that for all $h \in X := C([0,\pi])$ there exists a unique $u \in C^2([0,\pi])$ satisfying (0.7) and hence the map $K : h \to K(h)$ (is linear and) as an operator from X into itself is compact. It is also known that such a K has a sequence of positive, *simple* eigenvalues $\mu_1 > \mu_2 > \ldots > \mu_k \ldots$, such that $\mu_k \to 0$ as $k \to \infty$. Correspondingly, the linear Sturm–Liouville eigenvalue problem

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha\frac{\mathrm{d}}{\mathrm{d}x}u\right)+\beta u=\lambda u(x) \quad (0<x<\pi), \\ a_0u(0)+b_0u'(0)=a_1u(\pi)+b_1u'(\pi)=0, \end{aligned}\right\}$$

has a sequence of *simple* eigenvalues $\lambda_k = 1/\mu_k \to \infty$.

Another classical result we will need is the *Maximum Principle.*

Theorem 0.8 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $\lambda < \lambda_1$. If $u \in C^2(\Omega) \cup C(\overline{\Omega})$ is such that*

$$-\mathcal{L}u \geq \lambda u \text{ in } \Omega,$$
$$u \geq 0 \text{ on } \partial\Omega,$$

then $u \geq 0$ in Ω.

1

Differential calculus

This introductory chapter is mainly devoted to the differential calculus in Banach spaces. In addition to being a fundamental tool later on, the treatment of the calculus at this level permits better understanding at certain aspects, which might otherwise be neglected.
We discuss in Section 1 the Fréchet and Gâteaux derivatives as well as their elementary properties. The differentiability of the Nemitski operator is investigated in Section 2 and higher and partial derivatives are introduced in Sections 3 and 4, respectively.

1 Fréchet and Gâteaux derivatives

The Fréchet-differential is nothing else than the natural extension to Banach spaces of the usual definition of differential of a map in Euclidean spaces.

Let U be an open subset of X and consider a map $F : U \to Y$.

Definition 1.1 Let $u \in U$. We say that F is *(Fréchet-) differentiable* at u if there exists $A \in L(X, Y)$ such that, if we set

$$R(h) = F(u + h) - F(u) - A(h),$$

there results

$$R(h) = o(\|h\|), \tag{1.1}$$

that is

$$\frac{\|R(h)\|}{\|h\|} \to 0 \quad \text{as } \|h\| \to 0.$$

Such an A is uniquely determined and will be called the *(Fréchet) differential* of F at u and denoted by

$$A = dF(u).$$

If F is differentiable at all $u \in U$ we say that F is differentiable in U.

Hereafter, when there is no possible misunderstanding, *Fréchet differentiability* will be referred to simply as *differentiability*. A few comments on the preceding definition are in order.

(i) Let us verify that A is unique. Supposing the contrary, let $B \in L(X,Y)$ satisfy Definition 1.1 and $A \neq B$. It follows that

$$\frac{\|Ah - Bh\|}{\|h\|} \to 0 \quad \text{as } \|h\| \to 0. \tag{1.2}$$

If $A \neq B$ there exists $h^* \in X$ such that $a := \|Ah^* - Bh^*\| \neq 0$. Taking $h = t\,h^*$, $t \in \mathbb{R} - \{0\}$, one has

$$\frac{\|A(th^*) - B(th^*)\|}{\|th^*\|} = \frac{\|Ah^* - Bh^*\|}{\|h^*\|} = \frac{a}{\|h^*\|},$$

a constant, in contradiction with (1.2).

(ii) If F is differentiable at u then

$$F(u+h) = F(u) + \mathrm{d}F(u)h + o(\|h\|)$$

and F is continuous at the same point. Conversely if $F \in C(U,Y)$ then it is not necessary to require in Definition 1.1 the continuity of A. In fact (1.1) yields

$$A(h) = F(u+h) - F(u) - o(\|h\|)$$

and the continuity of F implies the continuity of A.

(iii) The definition of differentiability depends not on the norms but on the topology of X and Y only. That is if, for example, $\|.\|$ and $|||.|||$ are two equivalent norms on X then F is differentiable at u in $(X, \|.\|)$ if and only if F is in $(X, |||.|||)$ and the differential is the same.

Remark 1.2 The preceding comment (iii) could suggest the idea of extending the notion of Fréchet differentiability to locally convex topological spaces. The most natural way would be the following: let the topology of X (respectively Y) be induced by an infinite family of seminorms $|.|_{X,i}$ (resp. $\|.\|_{Y,j}$); define the differential of F as the linear continuous map A with the property that for all $|.|_{Y,j}$ there exists a seminorm $|.|_{X,i}$ such that $|F(u+h) - F(u) - Ah|_{Y,j} = o(|h|_{X,i})$. With such a definition all the main properties of the differential (below) hold true. Unfortunately, in dealing with the higher derivatives, there are